SYNTACTIC COMPLEXITY OF MUSIC WITH LAMBEK CALCULUS

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ABSTRACT

Although music and language differ in many respects, both exhibit hierarchical structure — notably between harmonic progressions and syntactic constructions. In this paper, we apply Lambek Calculus, a Gentzen-style sequent calculus used in categorial grammar, to the analysis of chord sequences. We first show how Lambek Calculus can model harmonic relationships in music through the lens of syntax and logical semantics, where the structure of a proof parallels that of a syntactic tree. Next, to demonstrate the feasibility of this approach, we assess the harmonic complexity of music by examining the proof tree: specifically, by counting both the number of tonal modulations and the depth of the derivation. This method allows us to explore whether intuitively optimal harmonic analyses correspond to minimal proofs within the system.

1. INTRODUCTION

Music and language are often said to share a common origin [1, 2], in that music could have been a primitive communication method of old human species. Accordingly, many music researches have been made with the linguistic theory. The generative grammar is a representative of such theory, where a finite set of generative rules decides a set of sentences, and in the similar way, generative rules have been considered for the sequence of harmonic chords [3–6]. In these days many approaches employ grammar-based systems as well as statistical parsing with annotated corpora [7–9], whereas the generative power of formal grammar in music has not been fully elucidated.

In this study, we pursue the theoretical aspect of grammar in music. In order to clarify the issue, we employ the categorial grammar, which is translated directly into a formal proof system: the sequent calculus [10], instantiated as Lambek Calculus [11]. Thus far, jazz-chord sequence has been analyzed with the categorial grammar [12], however, they have not mentioned its logical aspect. We develop the framework of categorial grammar to Lambek Calculus, and enhances both interpretability and transparency in the analytical process [13–16].

Lambek Calculus enforces the directionality and the linear order by disallowing structural rules such as exchange, contraction, or weakening. This reflects the sequential and non-commutative nature of harmonic progressions, where the order of chords is essential and repetitions or permutations are not freely admissible. Moreover, Lambek derivations are explicit proofs: each analysis corresponds to a well-defined logical object, whose structure is systematically studied and is not readily available in the traditional context-free grammar frameworks.

The calculus therefore provides a logically grounded, compositional account of chord sequences, with the additional advantage of being extensible through modal operators to represent modulations and tonal accessibility. While this study focuses on tonal music, the same formalism can be extended to modal and non-tonal systems — an aspect we aim to develop further.

To demonstrate the feasibility of this approach, we propose a measure of harmonic complexity based on the structure of the proof tree: specifically, the number of tonal modulations and the derivational *depth* of the proof. This allows us to introduce quantitative measures such as minimality and ambiguity. We argue that these metrics correlate with intuitively optimal analyses.

The paper is structured as follows: Section 2 introduces the basics of categorial grammar and Lambek Calculus, together with Modal Logic. Section 3 defines modal operators for tonal accessibility. In Section 4, we show an analytical example of *In Your Own Sweet Way* with our method. In Section 5, we further develop the advantage of proof theory, to measure the complexity of chord progression. Section 6 applies the proof depth to the jazz standard *All the Things You Are*. Section 7 discusses implications and outlines directions for future research.

2. GRAMMAR AND CALCULUS

In this section, we provide fundamental theories of language and logic.

2.1 Categorial Grammar

Categorial grammar (CG) [17] consists of a set of categories, assigned to each lexical word. A category is constructed recursively from a given basic categories, with the following (/) and (\); let X and Y be categories, then [1]

$$Y/X: \text{biting } X \text{ from the right-hand side, produces } Y,$$
$$Y\backslash X: \text{biting } Y \text{ from the left-hand side, produces } X.$$

With these categorial construction, we can argue if a given sequence of categories is *grammatical*, dependent on if we

[1] Note that (/) does not have any musical meaning tipically used in chord notation.

can finally obtain sentence category S, e.g.,

$$Y,\ Y\backslash(S/X),\ X \Longrightarrow S/X,\ X \Longrightarrow S.$$

A human language is said to belong approximately to the class of context-free grammar (CFG) in the Chomsky hierarchy [18] though there are known exceptions. [2] The generative power of CG is equivalent to CFG, shown as follows. A set of production rules of CFG can be rewritten in Chomsky normal form [19], each of which produces two non-terminal symbols (Y, Z below) or one terminal symbol (w) from a non-terminal symbol (X) as below

$$X \to Y\ Z, \quad X \to w,$$

and the branching in a syntactic tree becomes always binary except for terminal symbols at the leaves. The nodes in the tree are non-terminal symbols of CFG, which correspond to categories in CG. We can replace each CFG rule $X \to Y\ Z$ either for $Z = Y\backslash X$ or for $Y = X/Z$, and the resulting set of categories becomes equivalent to CFG.

For example, a verb phrase (VP) bites a subject noun phrase (NP) from left to be S, VP = NP\S. Also, a determiner (Det) bites a noun from right to be NP, Det = NP/N. Thus, we can compose a sentence as follows.

$$\dfrac{\dfrac{\underset{\text{NP/N}}{\text{A}} \qquad \underset{\text{N}}{\text{bird}}}{\text{NP}} \qquad \underset{\text{NP}\backslash\text{S}}{\text{flies}}}{\text{S}}$$

2.2 Lambek Calculus

We now translate the categorial constructions into a Gentzen-style sequent calculus. The turnstile symbol ($\vdash$) is read as "derives," and commas (,) on the left-hand side denote logical conjunctions ($\wedge$). In Lambek Calculus, the left-hand side of ($\vdash$) is specifically an *ordered* sequence of categories, and in accordance with intuitionistic logic, the right-hand side of ($\vdash$) is restricted to a single term [20].

We employ lowercase Latin letters ($x, y, z, \ldots$) to denote individual formulas, while uppercase Greek letters (e.g., Γ, Δ, Σ) represent sequences (or multisets) of formulas on the left-hand side of a sequent. Figure 1 summarizes the inference rules used to construct derivations. The expression y/x denotes a chord y that expects chord x to its right—i.e., y precedes x in the sequence. Conversely, $x\backslash y$ has the same interpretation but with reversed directionality: x expects y to its left. These rules can be applied either on the left side of the sequent ($\backslash_L$, $/_L$) or on the right ($\backslash_R$, $/_R$).

The *Cut* rule is a fundamental principle in proof theory, enabling compositional reasoning across intermediate formulas. However, in this paper, it is included for completeness but never invoked in actual derivations.

For example, consider the sentence "A bird flies." In categorial grammar, this can be expressed as a derivation using NP/N, N, and $NP\backslash S$. This construction is translated into the sequent calculus, providing horizontal lines step-by-step, as follows:

$$\frac{\Delta, y, \Sigma \vdash z \qquad \Gamma \vdash x}{\Delta, y/x, \Gamma, \Sigma \vdash z}\ (/_L)$$

$$\frac{\Gamma \vdash x \qquad \Delta, y, \Sigma \vdash z}{\Delta, \Gamma, x\backslash y, \Sigma \vdash z}\ (\backslash_L)$$

$$\frac{\Gamma, x \vdash y}{\Gamma \vdash y/x}\ (/_R) \qquad \frac{x, \Gamma \vdash y}{\Gamma \vdash x\backslash y}\ (\backslash_R)$$

$$\frac{\Gamma \vdash x \qquad x \vdash y}{\Gamma \vdash y}\ \text{(Cut)} \qquad x \vdash x\ \text{(Init)}$$

Figure 1. Set of rules of Lambek Calclus; we exclude ($\cdot$) rules.

$$\dfrac{\dfrac{\text{NP} \vdash \text{NP} \qquad \text{N} \vdash \text{N}}{\text{NP/N, N} \vdash \text{NP}}\ (/_L) \qquad \text{S} \vdash \text{S}}{\text{NP/N, N, NP}\backslash\text{S} \vdash \text{S}}\ (\backslash_L)$$

In this example, each category is introduced by an axiom and combined through inference rules, forming a syntactic proof that corresponds to the grammatical derivation.

2.3 Modal Logic

Modality is variously interpreted in music [21], and especially modal extension of Lambek Calculus [22] is useful to capture modulation in music, so we introduce a modal operator '$\Box$' by modal logic [23].

Modal logic is classically interpreted via Kripke semantics, which assumes a set of possible worlds connected by an accessibility relation R. The valuation (true or false) of a proposition depends on each world. A proposition P holds in a world w is written as $w \Vdash P$, and $\Box P$ is interpreted as:

$$w \Vdash \Box P \iff \forall w'\ (\ _{w}R_{w'})\ w' \Vdash P$$

where $_{w}R_{w'}$ states that w' is accessible from w. Namely, $\Box P$ holds in w if P holds in all the accessible worlds w' from w. Usual modal logic provides also $\Diamond$-operator as the dual of '$\Box$' as $\Diamond = \neg\Box\neg$, however, since Lambek Calculus does not possess negation ($\neg$) we do not include '$\Diamond$' in our system.

The '$\Box$' is introduced by the following axiom (K) in sequent calculus.

$$\frac{\Gamma \vdash x}{\Box\Gamma \vdash \Box x}\ \text{(K)}$$

In linguistics, $\Box$-operator can represent hypothetical contexts (e.g., conditionals or beliefs). A clause that has type S may be embedded as '$\Box S$' in another sentence, meaning it is evaluated in a different context. In our framework, we treat musical keys as possible worlds and employ '$\Box$' to refer to another key, as is shown in the following section.

3. LAMBEK CALCULUS IN TONAL PITCH SPACE

Throughout the paper, chord symbols appear in italics (e.g., Cm, $G7$), while key names are upright. Major keys are in

[2] For example, agreements in Romance languages or verb order in clauses in Dutch are context sensitive.

uppercase (e.g., C), minor keys in lowercase (e.g., a). Roman numerals indicate scale degrees: uppercase for major (I, II, ...) and lowercase for minor (i, ii, ...). The slash is equivocally used in music, as C/G, I/C, or V/V, meaning the base note in chord symbol, the degree in a key, or doppel dominant, respectively. In this paper, The slash symbol (/) is used only as a right-associative function in categorial grammar, avoiding ambiguous music notations.

3.1 Multiple accessibility

Given a set of diatonic notes, or a chord symbol, we can assign a degree within a key in different ways. We regard the key is a possible world in Kripke semantics and write $\varphi \Vdash x(c)$, where φ is a key, x is a degree, and c is a set of notes. For example, a Berklee chord symbol C is I in C major, IV in G major, V in F major, and so on.

$$\text{C} \Vdash \text{I}(C) \iff \text{G} \Vdash \text{IV}(C) \iff \text{F} \Vdash \text{V}(C).$$

This means that we may interpret a chord in a modulated context.

From now on, we employ white bold letters for modulation to related keys.

- $\mathbb{D}$ denotes modulation to the dominant key,
- $\mathbb{S}$ denotes modulation to the subdominant key,
- $\mathbb{R}$ denotes modulation to the relative major or minor,
- $\mathbb{P}$ denotes modulation to the parallel key (same tonic, opposite mode).

Figure 2 illustrates the map of related keys of western tonal music, based on the Tonal Pitch Space (TPS) [24]. [3] Thus, we introduce *multiple* accessible relations in our modal logic. We use $R_{\mathbb{D}}$, $R_{\mathbb{S}}$, $R_{\mathbb{P}}$, and $R_{\mathbb{R}}$ to represent the accesses to the above related key, respectively.

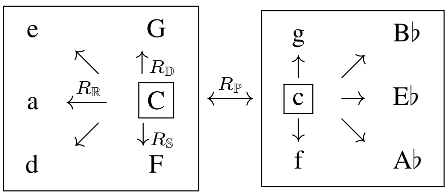

Figure 2. Accessible keys in the Tonal Pitch Space

In accordance with the four accessibility, we may introduce four modal operators $\{\Box_{\mathbb{D}}, \Box_{\mathbb{S}}, \Box_{\mathbb{R}}, \Box_{\mathbb{P}}\}$, however, we directly use $\{\mathbb{D}, \mathbb{S}, \mathbb{R}, \mathbb{P}\}$ for modal shifts for visibility, and '$\Box$' is employed only when it represents one of these modal shifts.

According to the conventional notation of modal logic, we should write

$$\varphi \Vdash \Box x(c) \iff R\varphi \Vdash x(c),$$

however in our case, since the access relation between tonalities is one-to-one, there exists such $\Box'$ that

$$\varphi \Vdash x(c) \iff R\varphi \Vdash \Box' x(c).$$

[3] Similar ideas to the TPS are also found in [25–27].

Therefore, we may attach the modal operator on the proposition in the destination world, for easy comprehension. For example, since G major is the dominant key of C major, we may write the equivalent formula, as follows.

$$\text{C} \Vdash \text{I}(C) \iff R_{\mathbb{D}}\text{C} \Vdash (\mathbb{D}\text{I})(C).$$

Later in Section 3.4, we will develop the sequent calculus with these modalities: let Γ be a sequence of chord degrees, and x be a chord degree. Then, axiom (K) is written, for example, as

$$\frac{\Gamma \vdash x}{\mathbb{D}\Gamma \vdash \mathbb{D}x}\ (\text{K})$$

and so on.

Note that $\mathbb{D}, \mathbb{S}, \mathbb{P}$, and $\mathbb{R}$ distinguish scale degrees, while $R_{\mathbb{D}}, R_{\mathbb{S}}, R_{\mathbb{P}}$ and $R_{\mathbb{R}}$ refer to modulations between keys.

These modal annotations enrich the expressive power of Lambek derivations. Rather than flattening all harmonic functions into a single key, we enable layered reasoning across multiple tonal centers. This is especially useful in jazz, where temporary tonicizations and frequent modulations are structurally significant. We also adopt a simplification principle: when multiple derivations exist, the one with the fewest modal transitions (or the shallowest modal nesting) is preferred. This principle of minimality complements the proof depth metric introduced in the next section.

3.2 Relation Calculi

We summarize the relations among modalities in Table 1, where R represents key-to-key accessibility and φ denotes a key.

$$R_{\mathbb{D}}R_{\mathbb{S}}\varphi = R_{\mathbb{S}}R_{\mathbb{D}}\varphi = \varphi \quad (1)$$

$$R_{\mathbb{D}}^{12}\varphi = R_{\mathbb{S}}^{12}\varphi = \varphi \quad (2)$$

$$R_{\mathbb{R}}^{2}\varphi = \varphi,\ R_{\mathbb{P}}^{2}\varphi = \varphi \quad (3)$$

$$\begin{cases} R_{\mathbb{D}}R_{\mathbb{P}}\varphi = R_{\mathbb{P}}R_{\mathbb{D}}\varphi \\ R_{\mathbb{S}}R_{\mathbb{P}}\varphi = R_{\mathbb{P}}R_{\mathbb{S}}\varphi \\ R_{\mathbb{D}}R_{\mathbb{R}}\varphi = R_{\mathbb{R}}R_{\mathbb{D}}\varphi \\ R_{\mathbb{R}}R_{\mathbb{S}}\varphi = R_{\mathbb{S}}R_{\mathbb{R}}\varphi \end{cases} \quad (4)$$

Table 1. Relations in accessibility

In Table 1, (1) expresses that the dominant of the subdominant (or vice versa) returns to the original key. Equation (2) shows that applying the dominant or subdominant modulation 12 times brings us back to the starting key, reflecting the circle of fifths in equal temperament. The relations in (3) show that parallel and relative modulations are involutive. Finally, the commutativity relations in (4) describe symmetric transitions between modulations.

In fact, these operators are not all strictly necessary: for example, we could use just one modality to track motion clockwise or counter-clockwise on the circle of fifths. Since $R_{\mathbb{D}}^{-1}\varphi = R_{\mathbb{S}}\varphi$, we can write:

$$R_{\mathbb{D}}^{m}\varphi = R_{\mathbb{S}}^{n}\varphi \quad \text{where } m + n \equiv 0 \mod 12.$$

Similarly, one can express a bridge between the major and minor circles with:

$$R_{\mathbb{P}}\varphi = R_{\mathbb{S}}^3 R_{\mathbb{R}}\varphi.$$

Nevertheless, for the sake of musical clarity and interpretability, we retain all four modal operators explicitly in our system.

3.3 Degree Calculi

The key modulations are applied here to degree calculation. We regard the degrees of chords, as well as those headed by the modal operators of $\{\mathbb{D}, \mathbb{S}, \mathbb{P}, \mathbb{R}\}$, as predicates for a set of notes in a key, that is a possible world.

In general, the x-th degree in key φ is equivalent to the $(x+3)$-rd degree in $R_{\mathbb{D}}\varphi$:

$$\varphi \Vdash x(c) \iff R_{\mathbb{D}}\varphi \Vdash (\mathbb{D}x)(c) \;\; (\mathbb{D}x = x + 3 \mod 7)$$

where φ: a key, x: a Roman numeral, and c: a chord symbol.

In addition, we need to distinguish upper-case degree numerals (major chords) from lower-case ones (minor chords). The full calculations are shown in Table 2. The first three can be considered the main ones, whereas the fourth is derived, since the Subdominant function can be derived from the Dominant function, and the Relative function can be derived from the Parallel one. Here, M represents a major key, and m a minor key.

$\mathbb{P}$	key, deg_M	=	key, deg_m
$\mathbb{D}$	key, deg_M	=	$\mathbb{D}\{key\}, \{deg + 3 \quad mod7\}_M$
$\mathbb{D}$	key, deg_m	=	$\mathbb{D}\{key\}, \{deg + 3 \quad mod7\}_m$
$\mathbb{S}$	key, deg_M	=	$\mathbb{S}\{key\}, \{deg + 4 \quad mod7\}_M$
$\mathbb{S}$	key, deg_m	=	$\mathbb{S}\{key\}, \{deg + 4 \quad mod7\}_m$
$\mathbb{R}$	key, deg_M	=	$\mathbb{R}\{key\}, \{deg + 5 \quad mod7\}_m$
$\mathbb{R}$	key, deg_m	=	$\mathbb{R}\{key\}, \{deg + 2 \quad mod7\}_M$

Table 2. Summarizing table for functions.

3.4 Tagged Sequent Calculus

We have now introduced the essential components of our formal system. In the Lambek Calculus notation, Γ, Δ, and Σ represent sequences harmonic contexts, while z stands for another chord formula in our system. We formalize modulations as transitions between possible tonal worlds, using modal operators to encode harmonic motion across key regions.

The (Init) rule represents the initial axiom or base case of a derivation, typically corresponding to the harmonic interpretation of a chord within a given tonality. For example, G $\Vdash$ IV(C) is shown as

$$\overset{C}{\text{G: IV} \vdash \text{IV}}$$

where we omit a horizontal line, to distinguish these initial introductions from other logical derivations.

In front of the sequents we proceed to use a key with colon (:) for mnemonic annotations, and call *Tagged sequents*, to remember the key in which we are doing our analysis. To avoid confusion, throughout this paper, we will always indicate the function of the key ($R_{\mathbb{D}}$, $R_{\mathbb{S}}$, $R_{\mathbb{P}}$, and $R_{\mathbb{R}}$) alongside the keys, even when they can be interpreted as belonging to a new tonal region. For instance, $R_{\mathbb{D}}$G corresponds to D, while $R_{\mathbb{D}}R_{\mathbb{R}}$d corresponds to C major. The key is valid for the entire line, not just for the subsequent sequent, and we will not repeat it if it is the same as the key in the previous line.

Example 1. Consider the example of $D7$–$G7$–$C\text{MA}^7$. When we interpret the sequence in C major, we obtain:

$$\dfrac{\dfrac{\overset{D7}{\text{C: II} \vdash \text{II}} \qquad \overset{G7}{\text{V} \vdash \text{V}}}{\text{II}, \text{II}\backslash\text{V} \vdash \text{V}}(\backslash_L) \qquad \overset{C\text{MA}^7}{\text{I} \vdash \text{I}}}{\text{II}, \text{II}\backslash\text{V}, \text{V}\backslash\text{I} \vdash \text{I}}(\backslash_L)$$

The final sequent i.e., II, II$\backslash$V, V$\backslash$I $\vdash$ I, can be interpreted as follows: in the tonality of C, we move from the second degree to the fifth, then to the first, which is also the last chord of the sequent, as can be seen on the right side.

Example 2. The flexibility of the system also enables the analyst to describe this type of tree in a different way, emphasizing the function of the *Doppel-dominant* as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{D7}{\text{G: V} \vdash \text{V}} \qquad \overset{G7}{\text{G: I} \vdash \text{I}}}{\text{V}, \text{V}\backslash\text{I} \vdash \text{I}}(\backslash_L)}{R_{\mathbb{S}}\text{G: } \mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\} \vdash \mathbb{S}\text{I}}(\text{K}_{\mathbb{S}})}{\mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\} \vdash \text{V}} \qquad \overset{C\text{MA}^7}{\text{I} \vdash \text{I}}}{R_{\mathbb{S}}\text{G: } \mathbb{S}\{\text{V}, \text{V}\backslash\text{I}\}, \text{V}\backslash\text{I} \vdash \text{I}}(\backslash_L)$$

In this case, the tagged sequents are rewritten in different tonalities because they change.

In some cases, modulations cannot be restricted to those with single operators; complex operators are also admissible.

Example 3. For example, $A7^{alt}$–$Dm7$– $G7$ is analyzed as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{A7^{alt}}{\text{d: V} \vdash \text{V}} \qquad \overset{Dm7}{\text{i} \vdash \text{i}}}{\text{V}, \text{V}\backslash\text{i} \vdash \text{i}}(\backslash_L)}{R_{\mathbb{R}}\text{d: } \mathbb{R}\{\text{V}, \text{V}\backslash\text{i}\} \vdash \mathbb{R}\text{i}}(\text{K}_{\mathbb{R}})}{R_{\mathbb{D}}R_{\mathbb{R}}\text{d: } \mathbb{D}\mathbb{R}\{\text{V}, \text{V}\backslash\text{i}\} \vdash \mathbb{D}\mathbb{R}\text{i}}(\text{K}_{\mathbb{D}})}{\text{VI}, \text{VI}\backslash\text{ii} \vdash \text{ii}} \qquad \overset{G7}{\text{V} \vdash \text{V}}}{\text{VI}, (\text{VI}\backslash\text{ii})/\text{V}, \text{V} \vdash \text{ii}}(/_L)$$

The sequent VI, (VI$\backslash$ii)/V, V $\vdash$ ii expresses that, in the key of C (i.e., the tonal center reached via the tagged sequent $R_{\mathbb{D}}R_{\mathbb{R}}d$), we first interpret a chord as the sixth degree, then move through a cadential progression that brings us to the second minor degree (ii), and finally to the fifth degree (V). This sequence reflects a typical jazz turnaround embedded in a modulated tonal context.

4. ANALYSIS (1) – *IN YOUR OWN SWEET WAY*

Now, we present an example analysis of the first eight bars of *In Your Own Sweet Way* by Dave Brubeck shown in Figure 3, and in Figure 6. To provide a clearer understanding, the main points are as follows:

- In the first tree, we track key shifts using shift modalities, namely, $\mathbb{R}$ (relative), $\mathbb{D}$ (dominant), and $\mathbb{S}$ (subdominant).
- Moreover, during the analysis, it is necessary to apply (1) to simplify $\mathbb{SDR}$ into $\mathbb{R}$. Indeed, it is easy to verify that g (G minor) is the relative key of B♭.
- The second tree presents an alternative interpretation of the same analysis. The final result remains unchanged, but a notable difference is that the cadence $Cm7$–$F7$–$B\flat^6$ is analyzed separately. Then, using the rule $(\backslash_L)$, we obtain the same result as in the first tree.
- The third tree (bottom left in Figure 6) is separate from the first one because it features a brief but significant key shift, which is better represented independently.
- The last tree (bottom right in Figure 6) illustrates a *ii–V–I* progression concluding the A-section of the piece. It is presented as a separate tree because it belongs to a different tonal region.

The final result will be a sequence of the three distinct regions of B♭ – G♭ – B♭.

5. DEPTH OF A PROOF

Now that the basic system is set, it is possible to introduce the notion of the depth of a proof in a way similar to how it is defined in Logic:

Definition 1 (Depth of a proof)**.** The depth of a proof is the maximum level of nesting in a proof, determined according to the rules in Figure 4.

The concept of depth allows us to more precisely determine how directly a proof represents the harmonic analysis of a given sequence of chords. A lower depth indicates a more immediate derivation, while a higher depth may result from additional structural elements, mainly modal operators, that clarify underlying harmonic relationships. Furthermore, the same set of chords can be analyzed in multiple ways, leading to different depth values, as demonstrated in Example 4. This highlights both the flexibility of the system in capturing different levels of harmonic interpretation and the shift in perspective that the analyst can choose to emphasize. In Figure 4 all the rules are described. The idea is that the depth increases whenever it is necessary to do an operation on the proposition, e.g., to write a cadence or to change the key.

Is is also possible to define what is *Minimality* in this context:

Definition 2 (Minimality of a proof)**.** A proof is *minimal* when its depth is lowest among all possible analyses of a sequence of chords.

Even though we cannot ensure that a given proof is minimal or not, it is possible to define the minimal proof as such that does not employ the axiom (K):

Definition 3. The minimality of a proof that does not require modal operators is the number of chords minus 1.

In fact, it is not difficult to see that, based on the rules in Figure 4, if we do not use the modal rules the depth can only increase every when we concatenate two different chords.

Example 4. This example presents the analysis of the chords (transposed to the key of F for practicality, so that the last chord is C7) that make up the B-part of *I Got Rhythm* (A7, D7, G7, C7). As is well known, this section consists of nothing more than a chain of dominants. Using Lambek Calculus, we provide two different analyses. The first one is the most direct: we simply write the degree of the chords in the key of C, resulting in the minimal possible depth of 3. The second analysis, instead, changes the key twice using modal operators ($\mathrm{K}_{\mathbb{S}}$). Although this increases the depth to 5, it makes the harmonic structure clearer, explicitly showing that the chords form a chain of dominants.

$$\dfrac{\dfrac{\dfrac{\overset{A7}{\mathrm{C{:}VI} \vdash_0 \mathrm{VI}} \quad \overset{D7}{\mathrm{II} \vdash_0 \mathrm{II}}}{\mathrm{VI}, \mathrm{VI}\backslash\mathrm{II} \vdash_1 \mathrm{II}}(\backslash_L) \quad \overset{G7}{\mathrm{V} \vdash_0 \mathrm{V}}}{\mathrm{VI}, \mathrm{VI}\backslash\mathrm{II}, \mathrm{II}\backslash\mathrm{V} \vdash_2 \mathrm{II}}(\backslash_L) \quad \overset{C7}{\mathrm{I} \vdash_0 \mathrm{I}}}{\mathrm{VI}, \mathrm{VI}\backslash\mathrm{II}, \mathrm{II}\backslash\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash_3 \mathrm{I}}(\backslash_L)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overset{A7}{\mathrm{D{:}V} \vdash_0 \mathrm{V}} \quad \overset{D7}{\mathrm{I} \vdash_0 \mathrm{I}}}{\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash_1 \mathrm{I}}(\backslash_L)}{R_{\mathbb{S}}\mathrm{D}{:}\ \mathbb{S}\{\mathrm{V}, \mathrm{V}\backslash\mathrm{I}\} \vdash_2 \mathbb{S}\mathrm{I}}(\mathrm{K}_{\mathbb{S}})}{\mathbb{S}\{\mathrm{V}, \mathrm{V}\backslash\mathrm{I}\} \vdash_2 \mathrm{V}} \quad \overset{G7}{\mathrm{I} \vdash_0 \mathrm{I}}}{\mathbb{S}\{\mathrm{V}, \mathrm{V}\backslash\mathrm{I}\}, \mathrm{V}\backslash\mathrm{I} \vdash_3 \mathrm{I}}(\backslash_L)}{R_{\mathbb{S}}R_{\mathbb{S}}\mathrm{D}{:}\ \mathbb{S}\{\mathbb{S}\{\mathrm{V}, \mathrm{V}\backslash\mathrm{I}\}, \mathrm{V}\backslash\mathrm{I}\} \vdash_4 \mathbb{S}\mathrm{I}}(\mathrm{K}_{\mathbb{S}})}{\mathbb{S}\{\mathbb{S}\{\mathrm{V}, \mathrm{V}\backslash\mathrm{I}\}, \mathrm{V}\backslash\mathrm{I}\} \vdash_4 \mathrm{V}} \quad \overset{C_{\mathrm{MA}}{}^7}{\mathrm{I} \vdash_0 \mathrm{I}}}{\mathbb{S}\{\mathbb{S}\{\mathrm{V}, \mathrm{V}\backslash\mathrm{I}\}, \mathrm{V}\backslash\mathrm{I}\}, \mathrm{V}\backslash\mathrm{I} \vdash_5 \mathrm{I}}(\backslash_L)$$

Finally we can see another tree with no modal operators that, maintains the same depth although the derivation is a little bit different from the first one that we have seen:

$$\dfrac{\dfrac{\overset{A7}{\mathrm{C{:}VI} \vdash_0 \mathrm{VI}} \quad \overset{D7}{\mathrm{II} \vdash_0 \mathrm{II}}}{\mathrm{VI}, \mathrm{VI}\backslash\mathrm{II} \vdash_1 \mathrm{II}}(\backslash_L) \quad \dfrac{\overset{G7}{\mathrm{V} \vdash_0 \mathrm{V}} \quad \overset{C7}{\mathrm{I} \vdash_0 \mathrm{I}}}{\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash_1 \mathrm{I}}(\backslash_L)}{\mathrm{VI}, \mathrm{VI}\backslash\mathrm{II}, \mathrm{II}\backslash\mathrm{V}, \mathrm{V}\backslash\mathrm{I} \vdash_3 \mathrm{I}}(\backslash_L)$$

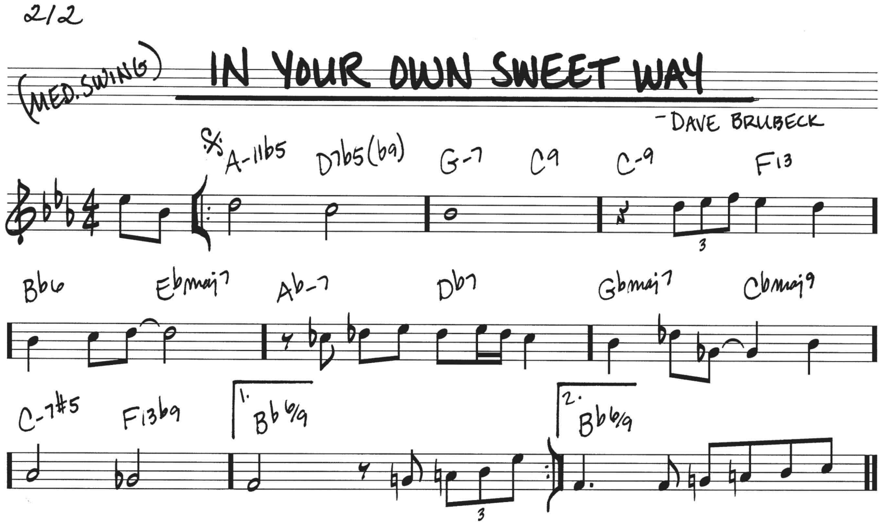

Figure 3. The beginning of *In your own sweet way* by Dave Brubeck

$$\frac{\text{Chord}}{x \vdash_0 x}\ (\text{Init})$$

$$\frac{\Delta, y, \Sigma \vdash_\delta z \qquad \Gamma \vdash_\gamma x}{\Delta, y/x, \Gamma, \Sigma \vdash_{\delta+\gamma+1} z}\ (/_L)$$

$$\frac{\Gamma \vdash_\delta x \qquad \Delta, y, \Sigma \vdash_\gamma z}{\Delta, \Gamma, x\backslash y, \Sigma \vdash_{\delta+\gamma+1} z}\ (\backslash_L)$$

$$\frac{\Gamma, x \vdash_\delta y}{\Gamma \vdash_\delta y/x}\ (/_R) \qquad \frac{x, \Gamma \vdash_\delta y}{\Gamma \vdash_\delta x\backslash y}\ (\backslash_R)$$

$$\frac{\alpha : \Gamma \vdash_\delta \Delta}{R_\Box \alpha : \Box\Gamma \vdash_{\delta+1} \Box\Delta}\ (K)$$

Where:

- $\delta, \gamma \in \mathbb{N}$;
- $K_\mathbb{F}$, $R_\mathbb{F}$ are respectively one of the modal functions applied and one of the modal relations;
- $\Box$ represents again each modality.

Figure 4. Rules of the Lambek Calculus for Chord Analysis with the calculus of the depths.

The idea of using depth arises from the fact that this number allows for comparing two different proofs and finding the one that is both the most explanatory for the analyst and the easiest to compute, thanks to the principle of minimality.

6. ANALYSIS (2) – *ALL THE THINGS YOU ARE*

All the things you are is a well-known song in the jazz repertoire, as was recorded by many celebrated artists, like John Coltrane, Keith Jarrett or Chet Baker (Figure 5). The interesting part of this song is the rapid succession of different keys and we can acquire various analyses thanks to Lambek Calculus in Figure 8. We should pay attention to the following interesting points about these analyses:

- The first point is that the initial tree analyzes a classic vii-ii-V-I progression that resolves to IV, after which there is a change in tonality. We have chosen to emphasize the fact that G7 functions as the VII major degree of $A\flat$, requiring multiple applications of the modal axiom (K) to reconnect the parts to the new key. However, we would also be able to divide the tree into two parts, providing a more specific analysis of the sudden modulation:

$$\dfrac{\dfrac{\dfrac{\vdots}{\text{vi}, \text{vi}\backslash\text{ii} \vdash_1 \text{ii}} \qquad \dfrac{\vdots}{\text{V}\backslash\text{I} \vdash_1 \text{I}}}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I} \vdash_3 \text{I}}\ (\backslash_L) \qquad \dfrac{D\flat\text{MA}^7}{\text{IV} \vdash_0 \text{IV}}}{\text{vi}, \text{vi}\backslash\text{ii}, \text{ii}\backslash\text{V}, \text{V}\backslash\text{I}/\text{IV}, \text{IV} \vdash_4 \text{I}}\ (/_L)$$

$$\dfrac{\dfrac{G7}{\text{C:V} \vdash_0 \text{V}} \qquad \dfrac{C\text{MA}^7}{\text{I} \vdash_0 \text{I}}}{\text{V}\backslash\text{I} \vdash_1 \text{I}}\ (\backslash_L)$$

In contrast, in the tree in Figure 8, we have grouped everything together for two reasons. First, D♭MA^{7}is

Figure 5. A part of *All the things you are*

closely related to $G7$, as they are separated by only a tritone. Second, the same pattern appears in the second tree. Ultimately, these differences reflect the main purpose of Lambek Calculus: to serve as a tool for the analyst, who can choose the approach that best fits their analyses.

- Another interesting point to note is that the results of the first tree in Figure 8 and the second tree are exactly the same in a different tonality, where they reveal the same harmonic structure transposed.
- Finally, in the last tree, it is possible to see that a similar mechanism is applied to the modulations, with the difference that from GMA^7, the modulation is to EMA^7—- only three steps on the circle of fifths – whereas in the rest of the cases, four steps were needed to connect the parts. This is also evident from the fact that the depth after the modulation increases by 3 instead of 4.

7. CONCLUSIONS AND FURTHER RESEARCH

Regarding the syntactic similarity between music and natural language, we have applied Lambek Calculus — corresponding to categorial grammar — to the analysis of chord sequences. Since the calculus is a formal proof system, it offers a robust and logically grounded framework. Moreover, it provides a clear and visual representation of harmonic structure that can aid music analysts.

In addition, we proposed a formal method to measure harmonic complexity in terms of the number of tonal modulations and the derivational depth of the proof tree. By quantifying these aspects, we introduced a notion of minimality that offers an optimality criterion for harmonic analyses.

Although we focused on tonal jazz harmony in this paper, the same formalism can be applied to other genres simply by modifying the modal and tonal systems. Since Lambek Calculus is inherently compatible with different rule sets, we regard our approach is highly adaptable for broader applications in music theory.

The current system, however, still includes the following issues. (i) We have added modal operators to access other keys into Lambek Calculus, however, we could not ensure the decidability and have not assessed the computational complexity by this extension. Although an inefficient algorithm requires $\mathcal{O}(n^3)$-time for parsing a sentence of length n, chord sequences are relatively short and thus less restrictive for practical applications. Also, different harmonic interpretations coexist in music, and thus, the notion of derivational ambiguity – the multiple proofs for a given sequence – may become conversely a meaningful measure of harmonic richness. (ii) For more rigorous introduction of modal operators, we would need the labeled sequent calculus where the labels (indices for possible worlds) represent keys; then, a modal operator works to access different label(s). Since the original Lambek Calculus do not include labels, we may need a big theoretical reconstruction. Instead of labels, we have tentatively employed 'tags' that is a tool outside of the formal syntax. Therefore, we need to adequately accommodate the tags into the sequent calculus.

Besides such logical issues, future directions include:

Automation A major objective is the development of computational tools capable of automatically producing harmonic analyses. Such tools would enhance the efficiency for both theorists and educators.

Semantic Interpretation Another crucial goal is the construction of a semantic framework to connect formal expressions with musical meanings — for example, mapping expressions like $V\backslash I$ to well-known concepts such as perfect cadences.

Empirical Applications Applying the system to large data sets of musical works will allow for empirical evaluation of its descriptive power and may reveal stylistic trends in harmonic practice. The concept of depth could serve as a meaningful parameter for comparing different genres or composers.

Generative Capabilities By reversing the analytic process, it will be possible to generate harmonic progressions from the system itself. Controlled variation of proof depth could guide the creation of musical material with desired levels of structural complexity — opening new avenues for computational creativity.

These developments aim to refine the theoretical foundation and expand the practical utility of Lambek Calculus in music analysis and beyond.

8. ACKNOWLEDGMENT

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Figure 6. Analysis of the first 8 bars of *In your own sweet way*

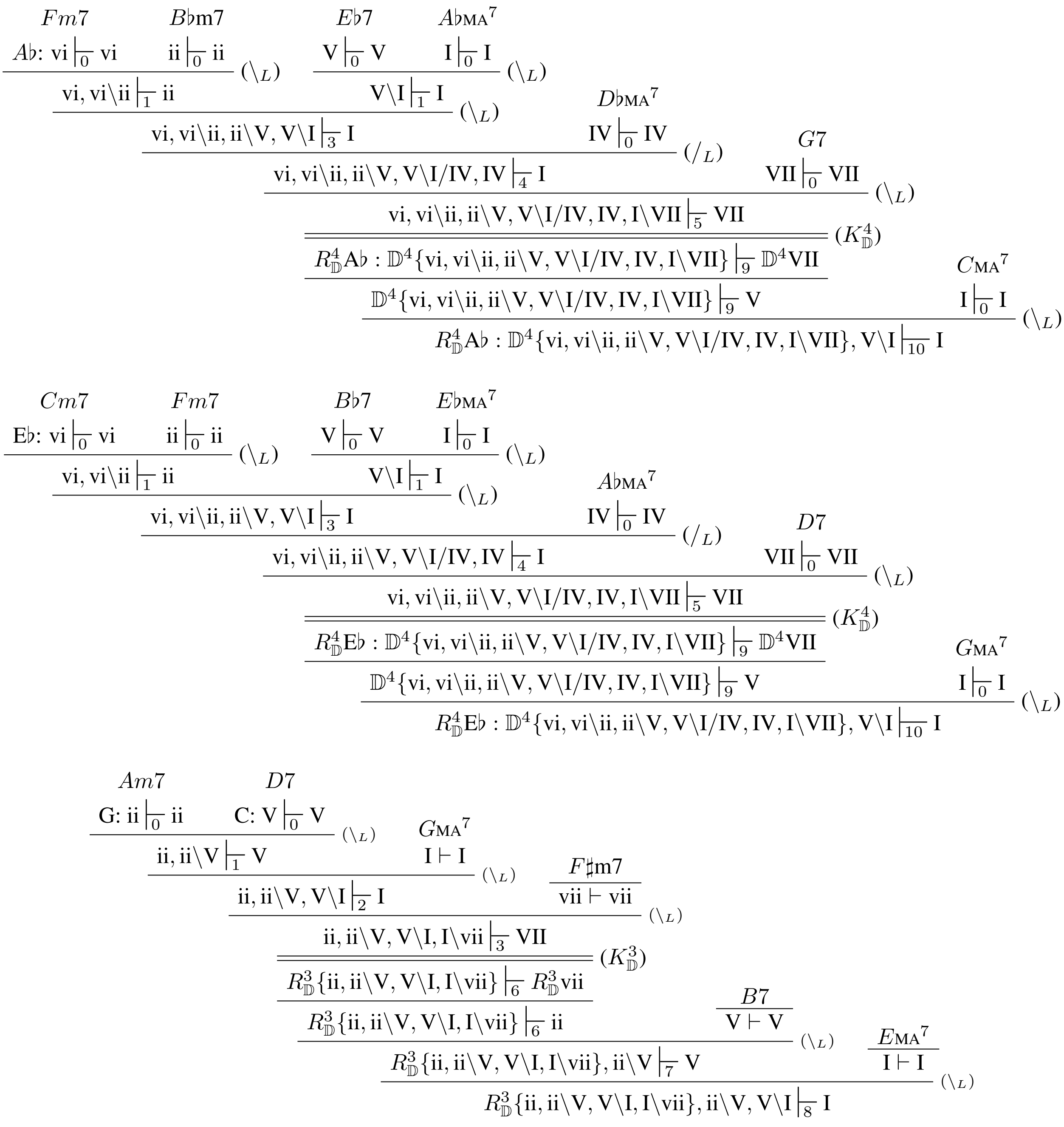

Figure 7. Analysis of the first 24 bars of *All the Things You Are* by Jerome Kern. Notice how the first two trees are exactly the same but in different tonalities. However, thanks to the $R^4_{\mathbb{D}}$ rule, it is also possible to highlight that the relationship between the two tonalities in both sections is identical. Without using this rule and instead attaching the new tonality directly—C in the first case and G in the second—the similarity between the two might be harder to perceive. The double lines indicate instances of the same rule.