MORTON FELDMAN'S "PROJECTIONS ONE TO FIVE" – EXPLORING A CLASSICAL AVANT-GARDE NOTATION BY MATHEMATICAL REMODELLING

Markus Lepper
semantics gGmbH, Berlin, DE
post@markuslepper.eu

Baltasar Trancón y Widemann
semantics gGmbH, Berlin, DE
Technische Hochschule Brandenburg a.d. Havel, DE

ABSTRACT

The compositions *Projection 1* to *Projection 5* by Morton Feldman are an important milestone in the application of graphical notation. The meta language `tscore` allows easy construction of a computer model of the original scores. On this model, automated performance, graphical rendering, and different analyses can be applied. The practical implementation work brings up the peculiarities of the original notational meta-model and scores, which without this effort are easily overlooked.

1. THE ORIGINAL SCORES

The compositions *Projection 1* to *Projection 5* were composed by Morton Feldman in 1950/51 and published by Edition Peters from 1959 to 1964. The format is that of a graphical score which only specifies the time position, duration, and the pitch register of the events, see Figure 1. The selection of the sounding pitches is left to the players. These works are among the earliest compositions with such "indeterminate pitches". "Feldman's graphic scores of the early 1950s are important [. . .] as works whose wide influence extended over the next few decades." [2, p. 10]

The project presented in this article takes a closer look to the syntax and the possible semantics of the "Projections". By constructing mathematical meta-models and realizing them as software, properties, prerequisites and problems become visible which can easily be overlooked otherwise, and indeed often have been. "[These works] have a certain iconic status in the popular and scholarly literatures, although very few have been analysed in close detail."[2, p. 10] Not before 2016 such more detailed analyses appeared [3].

Each piece is for a different selection of instruments, [1] but all share the same newly invented graphical score format, the intended sound character ("Dynamics are very low", except in no. 1), and even the tempo (BPM ≈ 72).

Despite its novel graphical appearance, the format indeed adheres widely to the conventions of standard notation, as voices appear vertically stacked and time flows from left to right. The staff for each instrument is a free space, into which non-overlapping rectangles are drawn, representing one played event each. According to the explanatory forewords (see below), the onsets and the offsets of these shall be read as aligned to quarters (= quarter notes = quarters of a box width), which are not explicitly represented visually.

The height of the event symbols is placed in the lower, middle, or upper third of the free space of the staff, indicating the duration for which an arbitrary pitch from the lower, middle, or upper sound register may be played.

So the base of the score is a fixed *grid*: Each measure has the same musical length (four pulses), and the same graphical width, and each page (beside possibly the last) holds the same number of measures, (Exception is no. 4 which changes between 10 and 11, see below.)

The measures are limited by vertical measure bars, which extend from the highest to the lowest staff which carries events in an adjacent measure. These bars are dotted lines and make up the vertical limits of the "boxes", mentioned as such in the explanatory forewords, but correspond to a conventional "measure". [2]

The events for the strings are further qualified as ordinary, harmonic, pizzicato, or sul ponticello. In nos. 1 and 4 only the first three appear and are distributed to three different staves. In no. 4 the solo violin has (in all three voices arco, pizzicato, and harmonics) few multi-stops. These are attributes with the number of pitches go be played, as are events in all piano voices. In all pieces, one of two piano staves is for audible key presses, an additional staff is for silently pressed harmonics. These shall resonate not only to the piano sounds but also to other ensemble instruments—the afterword to no. 2 says "Trumpet plays into open piano".

The staff for the audible piano events is the only one in which events may overlap horizontally, what happens seldomly (no. 2 ms. 56, 63, 65, 82; no. 3 ms. 7, 25; no. 4

[1] Determined by practical considerations, see [3, p. 18, 20]. Vigil states that each piece presents even a different *type* of ensemble. [1, p. 234] Feldman regarding the title: "My desire here was not to 'compose,' but to project sounds into time." ('Liner notes' in [4]) Boutwell sees a connection to Varèse [5, p. 465, 477]; Further considerations in [3, p. 24].

[2] These measures are for orientation only and do in no way imply metric emphasis or structure. [3, p. 92] In the open and extensible "LMN catalogue" of basic notation properties this can be described as NOTA.QUANDO.SEPARATORMETRIMENDAX or TEMPUS.COLORATUM.ABUSU. Currently covering only conventionally used Common Western Notation, LMN ([6], [7]) nevertheless lists about 890 different properties, defined by mathematical meta-models. These range from trivial facts to complex algorithms and are referred to by hierarchically structured identifiers, in analogy to zoology nomenclature etc. These are built on terms from the Latin language to restrict them to their role as mere labels for mathematical definitions and to avoid misleading connotations.

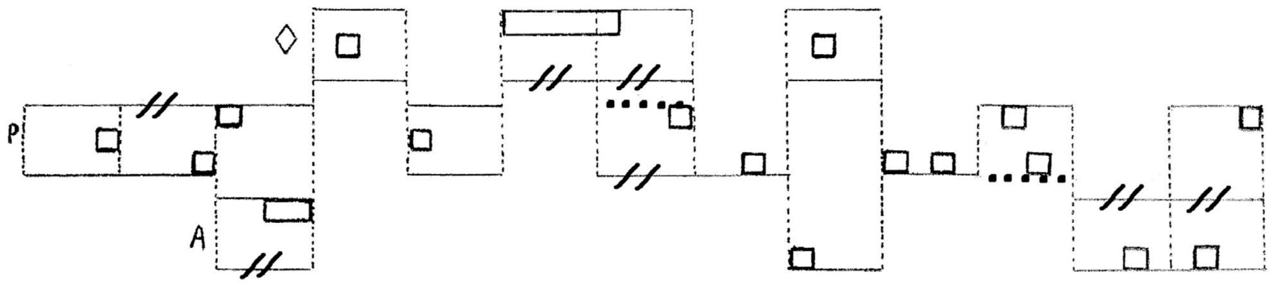

Figure 1. The first original page of Feldman, *Projection 1,* as printed by Peters, plus our correction marks "// " and "• • •".

ms. 25, 29. 33, 38, 44, 63, 64, 67, 77, 82; no. 5 ms. 2, 5, 17, 25, 34) and never with more than two sounds. [3]

In the work of Feldman, the usage of this style of graphic notation was only an intervening period, but an important one: It opened new ways of composing beyond the traditional focus on pitch classes. When the composer returned to exact notation, he had learned a much more free approach to pitch composition. "Feldman's conventionally notated music is himself playing his graph music", said John Cage, according to [8, p. 216].

2. ISSUES WITH THE ORIGINAL SCORES

The scores of all five works, as published by Peters separately, have some remarkable issues:

(Explicit definitions) Due to the radically new nature of the applied notation, this is defined explicitly by a short *explanation*, not more than one page, except for no. 4. So we have the rare and rewarding case that the semantics of notational devices are explained explicitly. [4] This page appears as prefix or postfix, is integrated into the page numbering or not, is written by hand or by type-writer, see Table 1.

(Edition errors) Proof-reading has been rather sloppy: The role of the figures with the piano notation (namely to give the number of keys to press simultaneously) is explained in nos. 3 to 5, but not in no. 2.

No. 2 is printed without the date of publication.

The title page of no. 5 says "for 3 Flutes, Trumpet, Piano and 3 Violoncelli", but indeed the score contains *two* pianos.

The staves in no. 5 are labelled with the instruments' names, except for the very first page. [5]

(Turning points) The layout of the scores is somewhat careless: Empty pages are interspersed in a way not to minimize the turning points but to maximize them.

(Empty bars) The score seems to be written by hand on a *transparent paper*, with a grid paper underlying and ruling the writing—in those days a common technique in architecture and engineering. But taking the grid away, which gives the look as it is printed, implies a *fundamental shift of paradigms*: Two adjacent empty measures (see no. 1, page 4, measures 48 and 49), and any empty measure adjacent to a line break, are not longer recognizable in a pure syntactic way! While everywhere else the sequence of measures is unambiguously recognizable even when written on rubber and arbitrarily deformed, [6] now the distances as such carry semantics.

Notably, the hand-written date of completion appearing at the end of all scores (except no. 5) does indeed *carry musical semantics*, because it indicates that no empty measures follow! [7]

(Horizontal lines) The horizontal separation lines get a very different treatment through the five pieces. This is discussed in detail in section 5.

3. REAL-WORLD PERFORMANCE ISSUES

With the piano: There are measures in which the pianist needs three hands! By automated data analysis (see section 7) we found that this is the case only in no. 2, measures 56, 63, 65, and 82. All these situations imply that there is one chord pressed in the "harmonic" piano staff, overlapping with two chords in the "sounding" staves. The occurring patterns are

```
 56              63/65            82
  YYYYY            YYYY           XXXX
XXXXXXX          XXXX             YYYY
  HHHHHHHH     HHHHHHHHHHH     HHHHHHHH
```
[= cases e)/g)   b)/g)   e)/g) from Table 2]

All patterns begin with some solo chord (sound or harmonics), so they can easily be realized with a *sostenuto pedal* (= "Steinway pedal"): This very first chord is taken into this pedal and the hand is free again.

If such a pedal is not available, the normal *sustain pedal* must be used instead. Of course this may only be active for the minimal possible interval: Short before the third chord must be played, that one of the sounding chords which has the earlier end ("XXXX" above) is taken into the pedal, which will be released with its end.

Table 2 shows all combinations for both kinds of pedals, especially the required silent returns. If the second and third chord come synchronously (like in ms. 82 and cases e) and f) in the Table) the first chord must be taken into the pedal anyhow. If it lasts longer than at least one of

[3] Found by automated analysis, see section 7. Discussed also in [3, p. 120p.].
[4] LMN property INFRA.REGULA.DICTA [6, p. 522].
[5] See also [1, Endnote 3].
[6] In the LMN catalogue this is the property NOTA.NONQUANTIBUS [6, p. 96].
[7] Cline discusses the role of an empty "box" at the end of the related work "Intersection I" (1951) [3, p. 176].

number	instruments	kinds of instruments	players	position of explanation	page number of expl.	medium of expl.	inter-staff gap	horizontal delimiter	measures per page	pages with measures	measures on last pg.	total measures	total duration at BPM 72	first staves labeled	subsequent staves labeled
1	Vcl	1	1	pre	–	hand	1	free	13	4	12	51	2’50	yes	no
2	Fl, Trp, Vl, Vcl, Pft	5	5	post	10	typewriter	0	free	10	9	5	85	4’43	yes	idem
3	2 Pft	1	2	post	–	typewriter	0	free	10	3	8	28	1’33	yes	idem
4	Vl, Pft	2	2	pre	I-II	hand	0.5; 2.5	reg	10; 11	8	–	84	4’40	yes	no
5	3 Fl, Trp, 3 Vcl, 2 Pft	4	9	pre	–	typewriter	0	solid	10	4	–	40	2’13	no[sic!]	yes

Table 1. Properties of *Projection 1* to *Projection 5*

$t_1 \leq t_2 \leq t_3 < u_1, u_2, u_3$

a)	$t_1 \leq t_2 < t_3$	$h_1 \neq h_2$	$p = t_3 - \epsilon$	$u_1 < u_2 \wedge u_1 \leq u_3$	$h_3 = h_1$	$p^* = u_1$	
b)	$t_1 \leq t_2 < t_3$	$h_1 \neq h_2$	$p = t_3 - \epsilon$	$u_2 < u_1 \wedge u_2 \leq u_3$	$h_3 = h_2$	$p^* = u_2$	
c)	$t_1 \leq t_2 < t_3$	$h_1 \neq h_2$	$p = t_3 - \epsilon$	$u_1 = u_2 \wedge u_1 \leq u_3$	$h_3 = h_x$	$p^* = u_2$	
d)	$t_1 \leq t_2 < t_3$	$h_1 \neq h_2$	$p = t_3 - \epsilon$	$u_3 < u_1 \wedge u_3 < u_2$	$h_3 = h_x$	$p^* = u_3$	$R_{x,3} = p^* - \epsilon$
e)	$t_1 < t_2 = t_3$	$h_2 \neq h_3$	$p = t_2 - \epsilon$	$u_1 \leq u_2 \wedge u_1 \leq u_3$		$p^* = u_1$	
f)	$t_1 < t_2 = t_3$	$h_2 \neq h_3$	$p = t_2 - \epsilon$	$u_y < u_1$		$p^* = u_y$	$R_{1,y} = p^* - \epsilon$
g)	$t_1 < t_2 \leq t_3$		$t_1 < q < t_2$			$q^* = u_1$	
h)	$t_1 = t_2 < t_3$		$t_1 < q < t_3$	$u_1 < u_2$	$h_3 = h_1$	$q^* = u_1$	
i)	$t_1 = t_2 < t_3$		$t_1 < q < t_3$	$u_2 < u_1$	$h_3 = h_2$	$q^* = u_2$	
j)	$t_1 = t_2 < t_3$		$t_1 < q < t_3$	$u_1 = u_2$	$h_3 = h_x$	$q^* = u_x$	

Explanation:

- t_n = the start timepoints of a chord, given by the index
- h_n = the hand used to play the chord with the same index
- u_n = the end timepoint of the chord with that index
- p = the timepoint of pressing down the (normal/sustain) pedal
- p^* = the timepoint of releasing the (normal/sustain) pedal
- q = the timepoint of pressing down the Steinway pedal
- q^* = the timepoint of releasing the Steinway pedal
- $R_{a,b}$ = the timepoint of returning silently to the chord number a, letting go chord number b
- $t - \epsilon$ = right before t
- $x \in \{1, 2\}$ $y \in \{2, 3\}$

Table 2. The different cases when playing three chords with only two hands.

the others (=case f), a hand must re-press its keys silently, immediately before the pedal is released, see Figure 2. A similar method must be applied when the third chord is shorter than all others (case d). These cases do not occur in the composition, but are likely when composing in this style.

The use of the normal pedal blurs the resonance effects intended with the “harmonics” voice. Such situations occur only in emphProjection 2, which is the first in the cycle which uses a piano. Apparently by the time the composer became aware of this effect and avoided it. [8]

With the strings: Programmed analysis shows that multi stops are only applied in the violin in no. 4. In the “harmonic” voice one double stop in ms. 32, in the “arco” voice on triple stop in ms. 17, both in the middle register. The “pizzicato” voice contains the most, which is appropriate to the nature of the tone production, namely two double, three triple and even one quadruple stop. Per register:

	2	3	4
hi		1	1
mid	1		
lo	1	2	

Problematic are the two triple stops in the low register: The lowest solution are the open string G3+D4+A4 (in MIDI notation), the top of which comes very close to any possible “mid” register.

The opposite problem with the quadruple stop in the high register: all four strings must be involved, over a distance of nearly two octaves of their tunings, which requires a wide definition of that register to allow the left hand to compress all pitches into it.

Comprehension: It appears as if the new way of composing was practised by Feldman in a primely more ab-

[8] In the subsequent works “Intersection 2 and 3” the piano part became again much more virtuoso, see [3, p. 40].

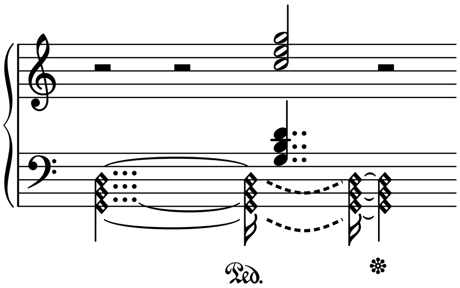

Figure 2. A method for playing overlapping piano events according to case f) in Table 2.

stract and programmatical way, not treating very thoroughly the issues of a concrete realization. [9] Since the idea of letting the pitches to the decision of the players was utmost revolutionary anyhow and dominated all considerations, this is understandable.

4. THE TSCORE MODEL

`Tscore` is a meta-meta-model for constructing models for the syntax and semantics of musical artefacts. [11] [12] [13] Figure 3 shows its simple fundamental paradigm in the style of entity–relationship. It may be surprising how well-known components of a model naturally exist on *different layers* of the architecture: The fundamental formula for representation (last line in the figure) is the core of the meta-meta-model. It states that every event • is identified by a pair of values: a voice from V and a timepoint from T. Each event relates to a collection of values from the parameter domains $D_a, D_b, \ldots$, indexed by parameter names from P. The events and the identity of the voices V change with every model, but all other types are defined with the meta-model—in `tscore` even the time structure is pluggable.

To encode and process all composition *Projection 1* to *Projection 5'*, only one single meta-model had to be developed, the instances of which are the models of these pieces. Table 3 shows the "timeless" parameters used to adapt the meta-model to the needs of the different pieces. [10]

The complete code for parsing and evaluating are 275 lines of code (measured with "cloc" [14], not including the `main(..)` wrapper and file reading). plus about 400 loc for the graphical and 300 lines for the acoustic rendering. Plus arbitrary many for wanted analyses. All code (except the score data itself) is in the public domain; a publication as open source is under preparation.

Figure 4 shows the beginning of the `tscore` source text of no. 2: The events are realized by the codes "`h`", "`m`", and "`l`", for the high, middle, or low register. The events in the voices "`vl`" and "`vc`" for the string instruments may carry an additional character as qualifier: "`h`" for harmonics, "`p`" for pizzicato, and "`P`", for sul ponticello. In nos. 1 and 4, only the first two are used—realized in the Peters score not by modifiers but by distributing the events to three separate systems. In no. 4 the solo violin may carry an additional digit to indicate multiple stops. The modifier "A" for arco/ordinario from the original score is realized in the `tscore` version as the default case = entered with no visible modifier. The piano voices must carry an additional decimal number of the keys to press.

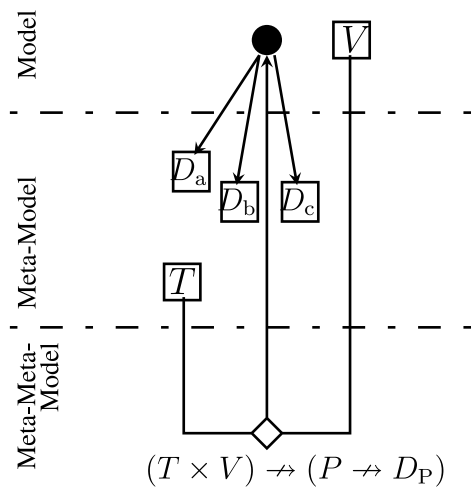

Figure 3. The fundamental paradigm of `tscore`: an event-based co-algebraic data model

The format of each voice is declared to the `tscore` parser by the timeless property `addenda`.

The rhythm is defined by (a) the division of the measure into 1, 2, or 4 entries, supported by (b) positive or negative dotting notation (see measures 3 to 6 of voice `fl`), plus (c) the rule that every event lasts up to the onset of its successor in the same voice. This may by a pause "`%`" or a prolongation "`-`". It took only one hour to enter the complete no. 2, including proofreading supported by the graphic output.

Since `tscore` defines "events" by a unique combination of "voice" and "timepoint", the piano staffs which contain timely overlapping events have been modelled by multiple voices (in fact: only two). Both voices are entered and processed separately, but written into the same staff / the same synthesizer channel when rendering. This is controlled by the timeless voice parameter `cumLinea`, see Table 3.

5. GRAPHICAL RENDERING OF THE MODELS

Our interest in the notation of Feldman's grid music was inspired by the Neoscore project, a new graphical music notation library [15] which used *Projection 2* as a test case. The idea suggested itself to employ a `tscore` model as the front-end, for more comfortably generating the data. Indeed the co-algebraic and visitor-based architecture of `tscore` allowed to implement the export into the Python data readable by Neoscore by just 40 lines of Java code—see Figure 5 for an example output of this co-operation.

The Neoscore rendering is only a proof-of-concept prototype, specially addressing no. 2. But by using `tscore`

[9] See also [9, p. 73] and the detailed discussion w.r.t "Untitled Composition" in [10, p. 33].

[10] The names have been chosen from the Latin language for smooth integration into the LMN catalog.

timeless parameters per score:
• `visuum.interVoces` : $\mathbb{Q}$
// = gap between staves, given as a factor of event height
• `visuum.tactiInPagina` : $\mathbb{N}$
// = number of measures on each page
// (in no. 4 the start of a new page is marked instead
// by an arbitrary event in the dedicated voice "`pg`".)
• `signumFinis` : String
// = date signature, indicates end of score

timeless parameters per voice:
• `addenda` : {**claves**, **arcus**, **arcusMult**}
// = type of the voice, thus the allowed event modifiers
• `cumLinea` : String
// = name of the voice with which the staff is shared
• `nomen.longum/breve` : String
// = the staff label for the first page / the subsequent pages
// (either these or `cumLinea` may be given.)
• `subordinatum`
// = marks the "piano harmonic" staves for special graphics

Table 3. Timeless parameters defined by the `tscore` "Projection_n" meta-model

it could easily be shown that Neoscore can deal with all five pieces. Nevertheless it addresses specially no. 2, and since all pieces differ also graphically (see again Table 1), a more versatile but ad-hoc rendering (393 loc) has been programmed in Java. This has been used to explore the rules and exceptions of the graphical design systematically.

Which properties must be included and which may be neglected is the central question when remodelling the graphical appearance of the scores by Feldman—as already discussed above for the end notes. For instance, Wiener regrets "[daß] die späten Werke [. . .] nicht mehr als Kopie der Feldmanschen Handschrift in den Handel kommen, sondern ein [. . .] anonymer Notensatz sie sowohl der [Seiten-] Umbrüche als auch der so wichtigen äquidistanten Taktstriche beraubt, kann nicht mehr nur ≫bedauerlich≪ genannt werden, denn hier ist eine ganze Sinnebene, vielleicht die tragende dieser Werke, besinnungslos zerstört worden." [11] [16]

The most significant issue which popped up concerns the *horizontal limiting lines*. As indicated graphically in the prefaces, the events in the high register are indicated by boxes "hanging from" the top line of their "measure box", the low register events "stands on" the bottom line, and the middle register events stand away from both. In most cases the horizontal lines in the score appear accordingly: the bottom measure box line with low register events, the top with the high register, and both with middle register events. [12]

In the original scores, Feldman adhered strictly to this "preface rule" only in no. 4. In no. 3 and 5 there is no gap between the staves, and all horizontal limiters extend over all measures (except for the middle page of no. 3 and the second half of no. 5). In nos. 1 and 3 the appearance of the horizontal limiters basically follows that preface rule, but in detail is irregular and seemingly arbitrary! In Figure 1 all horizontal lines are marked which are superfluous or missing according to that rule. [13]

The appearance of these horizontal limiting lines does not carry any semantics for the execution of the piece. Therefore our data model does not contain this information, and the implemented Java rendering algorithm strictly follows the preface rule, as the manuscript of no. 4 does. [14]

One possible interpretation could assign them *ergonomic use*, to make the synchronization of voices and/or the pitch registers more easily readable. [15] This indeed maybe the case in some particular score situations.

But seen as a whole, a different aspect seems more plausible: Knowing about the composer's interest in contemporary art, and his close contact and collaborations with Guston, Pollock, Rothko, Rauschenberg, and other painters, it is even more likely to give the graphical appearance an *aesthetic value on its own.* [16] "Painting and music [—] using them interchangeably or metaphorically in such an extent that the two art forms became one in much of Feldman's writing." [4, p. XIX] The horizontal and vertical lines can thus be seen as a direct echo of New York City's vertical and horizontal structures. [17] Seen from the top of the Empire State Building when sun sets, the selectively lit windows around follow closely the patterns of Feldman's event boxes.

6. SOUND RENDERING OF THE MODELS – AUTOMATED PERFORMANCE

6.1 General Considerations

The prefaces of all five pieces make no prescription at all about the pitches to be played. "What Feldman is assuming [. . .] is that the performer is a sensitive and inspired musician who has the best interests of the work at heart." [8, p. 217] These five pieces are doubtlessly among the very first in history in which pitch is not exactly specified by the composer, while onset, duration and dynamics are. John Cage, Earle Brown, and others worked at the same time in similar directions, and the true historic priorities will probably never be cleared. [18]

One possible way of interpreting the scores nowadays is an automated realization with today's computer technology. From this we expect insights in at least two con-

[11] "That the late works are no longer sold as copies of the manuscript, but in an anonymous computer rendering (which removes the page breaks and the important equidistance of the measure bars) is more than just ≫regrettable≪, because a whole layer of meaning, perhaps the fundamental one, has senselessly been destroyed." For a similar standpoint see [10, p. 4, 19].

[12] Cline states that the vertical location "is always apparent, without recourse to a ruler", means: to measuring. [3, p. 76]. Indeed it is more, namely NOTA.NONQUANTIBUS = purely syntactic = resistant to any deformation.

[13] According to [3, p. 22, 34], John Cage was involved in finding the final appearance.

[14] Two derivations from that preface rule are not taken over by our implementation:

- The larger gap and the break of the measure bars between violin and piano in no. 4.
- The aligned start of all staves and the uninterrupted horizontal staff lines (with apparently irregular exceptions) in nos. 3 and 5.

[15] Property NOTA.FACTUM.LEGERESIMPLIFICANS in [6, p. 18].

[16] Property NOTA.UTPICTURA.SIGNIFICANS in [6, p. 534].

[17] For grids as emblematic for modernism see [3, p.80,81].

[18] "John Cage was primarily interested in those aspects of Feldman's music that he understood as 'indeterminate'."[2, p. 12] Cline lists American predecessors who composed indeterminate works and discusses a possible precedence of Earl Brown [3, p. 3, 15, 39].

```
PARS sola
visuum.tactiInPagina = 10   visuum.interVoces = 0
signumFinis  = "Jan 3, 1951"
VOX fl                  nomen.longum = "Flute"   nomen.breve = "Fl."
VOX tr                  nomen.longum = "Trumpet" nomen.breve = "Trp."
VOX vl addenda = arcus  nomen.longum = "Violin"  nomen.breve = "Vl."
VOX vc addenda = arcus  nomen.longum = "Cello"   nomen.breve = "Vc."
VOX p1 addenda = claves nomen.longum = "Piano"   nomen.breve = "Pn."
VOX p2 addenda = claves cumLinea = p1
VOX ph addenda = claves nomen.longum = "◇"       nomen.breve = "◇"   subordinatum = est

T      1     2       3           4         5         6        7      8         9      10       11
VOX fl %             %.      h   -.     %            m .%     %.   l %         %.   l -   %
VOX tr %  m  -  .%   l - %  h   %                             %.   h %         %.   l -   %
VOX vl %             % hh %  %   %.     m  - .%                      %.     mp %.   m -   %
VOX vc %             % %  mp %   %  % l % %  %         % l hh % %      % % mp %  %.   m - - h  %
VOX p1 %     m1 .%   % %  l1 h7  %         % l2 % %           %.  m3 %              %.       l1
VOX p2 %
VOX ph % l5. -       -    %                          %   m1   -.  %
```

Figure 4. Timeless parameters and the first page of Feldman, *Projection 2*, as `tscore` source

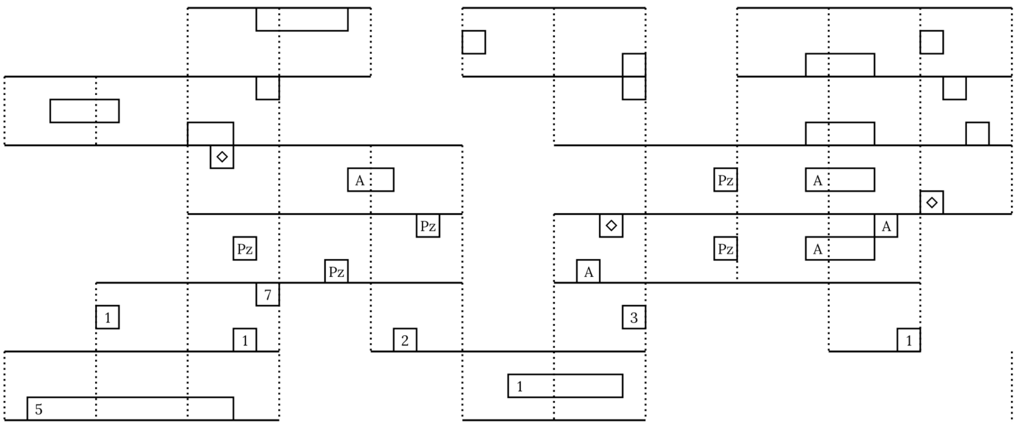

Figure 5. The first page of Feldman, *Projection 2*: data by `tscore`, rendered by Neoscore [15].

cerns: (a) Programming an algorithmic interpretation requires to take this "ancient" notation seriously—replacing the former improvisations by computer-based synthesis techniques is a kind of mathematical re-modeling. It will bring to light semantical problems and peculiarities which obviously the actors in the 1950s were not aware of, or did not care about. So our approach is a kind of "re-enactment" or "experimental archeology".

(b) Producing series of these realizations with different parameter settings can be used for empirical experiments about the effect on the listeners—a promising plan, but outside the current scope of our project.

With automated interpretation, the "sensitive performer" as requested by O'Hara is replaced by an algorithm which makes sensible transformations of the raw data delivered by an "insensible" random generator. For mimicking the improvisational behavior of "sensitive performers", the following aspects must be discussed and decided:

(A) Pitch class reservoir: A natural way for an ensemble of improvising musicians is to agree on a set of pitch classes to use throughout the performance. Since all pieces (beside no. 1) employ pianos, to play in the equally tempered tuning suggests itself. Anyway, it was the tuning system of the works of Webern and other composers of dodecaphonic serialism, the avant-garde in those times took

"onetone"—pitch class "c" only
"twotones"—pitch classes "c" and "d♭"
"bach"—pitch classes "a", "b♭, "b♮", and "c"
"tonic"— pitch classes "c", "e", and "g"
"blueNote— pitch classes "c", "e♭", "e♮", and "g"
"dominant"—pitch classes "c", "e", "g", and "b♭"
"pentatonic"—"d♭", "e♭", "g♭", "a♭", and "b♭"
"hypochromatic"— "c", "d", "e", "f♯", "g♯", and "b♭"
"dodecaphonic"—all twelve pitch classes

Table 4. Characteristic pre-selections of allowed pitch classes

as their starting point. [19] Beyond this, further restricting the set of allowed pitch classes can lead to quite different stylistic and atmospheric results—see Table 4 for typical examples. This set is defined as `PCStart` in Table 5.

(B) Definition of the three instrumental registers: In spite of their role as the fundamental datatype for the score constructions, the explanatory forewords are widely unspecific about the three instrumental registers:

(B1) It is not discussed how the registers shall be received by the listener: Shall the separation be recognized clearly

[19] For Cages's and Feldman's "mutual enthusiasm for Webern" see [5, p. 461].

as such? Or shall it only organize diffuse processes between lower and higher regions, which are perceived as a continuum? The range definitions are contained in Table 5 as the data $\texttt{range}(i)$.

The two integers in $\texttt{gaps}(i)$ are derived. Making registers explicit requires a considerable high positive integer therein. But the forewords do not even forbid *overlapping* registers = negative gaps values! We follow the first interpretative opinion and impose on our realizations the predicate $\texttt{wellSeparated}(i)$, with $i = 5$ as a mere matter of taste.

(B2) Naturally the high register of a lower instrument overlaps with the low registers of a higher one, w.r.t. absolute pitches, e.g. between violin and violoncello. Contrarily the lower registers in any instrument have, for physical reasons, a broader spectrum and a more sonorous timbre. By this, a low violin tone can be received as more related to a low cello tone than to a high one.

Again it is not explicitly said which alternative is intended, or whether there is any intention at all.

(B3) The ease of defining registers varies much with the kind of instrument. The predicate $\texttt{complete}(i)$ says that all pitch classes from PCStart are represented in every register defined for instrument i. Together with the predicate $\texttt{wellSeparated}(i)$ and the pitch class **dodecaphonic**, this requires a range of about four octaves—easily realizable by the piano, but hardly by any other instrument.

(C) Entropy and repetition:

Feldman reports about a performance in which "the players decide together, before the concert, actually to sabotage it – and they decided in this particular section they were going to play 'Yankee Doodle,' with the amount of notes called for and in the register in the score." [3, p. 45].

In general, Cline conjectures Feldman's "intense dislike for what he regarded as formulaic choices, reflecting conditioned responses or remembered sequences, [...] playing of scales or familiar [...] spoiled the performance for him."

When trying to formalize and automate these effects, both "sabotage" and "remembered sequences" can be seen as a *lack of entropy*, and a natural remedy is the ban of repetitions, similar to dodecaphonic techniques.

Those (as invented and propagated by Schoenberg, and even more in the very concentrated form practiced by Webern) were still the most recent avant-garde revolution when the *Projections* were composed. They were ideal and inspiration, but at the same time also already a tradition which Feldman and his colleagues tried to overcome.

Therefore it seems sensible to explore mathematically, how a combination of those older principles with the newly proposed register notation will work, for simulating the required "sensitive performer". For each of our realizations one of these rules must be selected:

$\texttt{eval}_{\text{Free}}$ = no restriction on repetitions of pitch classes

$\texttt{eval}_{\text{Instr}}$ = repetition not earlier than unavoidable, decided locally to each voice

$\texttt{eval}_{\text{Impro}}$ = the same, but decided globally for the whole ensemble, by improvisation

$\texttt{eval}_{\text{Comp}}$ = the same, but decided out-of-time, by "composing"

The rules $\texttt{eval}_{\text{Instr}}$ thru $\texttt{eval}_{\text{Comp}}$ correspond roughly to Schoenberg's "Wiederholungsverbot" (ban of repetition), which directly led to the invention of the "Reihentechnik" (dodecaphonic serialism). More precisely: When the interpretation starts, the selection of pitches aims to produce a *maximum-length segment without repeating pitch classes*. As soon as such a repetition becomes *unavoidable* (for what reason ever, see below!) a new such cycle is started. That means that with four elements, the initial cycle a–b–c–d can be followed by d–c–b–a, or by any other permutation.

This concretization of the rules $\texttt{eval}_{\text{Free}}$ thru $\texttt{eval}_{\text{Comp}}$ is sensible aesthetically, but also practically: With some training a musician can memorize which pitch classes from PCStart have been present in the current cycle (= played by themselves and maybe heard from the others) and when a new such cycle must begin.

However, the difference between $\texttt{eval}_{\text{Impro}}$ and $\texttt{eval}_{\text{Comp}}$ is significant: Whenever multiple players have to play events simultaneously, they cannot avoid possible doublings of pitch classes. [20] Therefore $\texttt{eval}_{\text{Impro}}$ and $\texttt{eval}_{\text{Comp}}$ are fundamentally different: $\texttt{eval}_{\text{Comp}}$ means "composing", when the translation from the graphical score into particular pitches happens "out of time", with full knowledge of the synchronous events. [21]

Furthermore, the key presses in the "harmonic" voices of the pianos cannot be heard by the other players, at least not immediately. Therefore we exclude them from our four pitch selection algorithms. [22]

6.2 Experimental Implementation

A main issue when realizing the pieces and defining the automated algorithm lies in the fact that during execution, aspects (A), (B), and (C) as described above are not independent but contrarily closely related: For instance, when an instrument shall play from a particular register and all pitch classes contained therein have already appeared, a new cycle must be started, as described above, in spite of still available pitch classes in other registers.

Table 5 shows the basic data types and table 6 the four different algorithms for the automated realization of the scores. O stands for the octave registers, in MIDI notation

[20] Of course one could establish additional communication channels between the players, beyond just playing and listening. This is not considered in this article.

[21] Such transcribing corresponds to the practical setting in Cage's *Music for Carillon (Graph) No. 1* [3, p. 37]. Further methods are sensible, e.g. that all players agree on one particular "series" = permutation of PCStart, which underlies the methods $\texttt{eval}_{\text{Instr}}$ to $\texttt{eval}_{\text{Comp}}$, either only verbatim or in its four modi (with retrograde, inverse, and retrograde-inverse). Or a series of such series may be defined. Or every player selects their own such series.

Or the set PCStart itself may change over the duration of the improvisation, which would also make the simple rule $\texttt{eval}_{\text{Free}}$ more interesting. All these variants may be left to future work.

[22] We must exclude them from the audible realization anyhow, as long as this is restricted to third-party off-the-shelf "general MIDI" instruments. Nor can "sul ponticello" be realized. Anyhow, also in real-world execution the "sympathetic resonances [are] often inaudible". [3, p. 21; detailed discussion p. 127]. Vigil even considers "more [...] theatrical than musical reasons." [1, p. 236]

// universal types and data:
$\mathsf{O} = \{-1, 0, \ldots, 8\}$ *// = octave registers*
$\mathsf{C} = \{0, \ldots, 11\}$ *// = all (enharmonic) pitch classes*
$\mathsf{P} = \mathsf{O} \times \mathsf{C}$ *// = all pitches*
$\mathtt{key} : \mathsf{P} \to \mathbb{N}$ *// = pitch to MIDI key*
$\mathtt{key}(o, c) = c + o * 12 + 12$
$p_1 < p_2 \iff \mathtt{key}(p_1) < \mathtt{key}(p_2)$
$\mathtt{random}[X] : \mathbb{P}X \to X$ *// = select froam a non-empty set*
$\mathsf{R} = \{\mathtt{lo}, \mathtt{mid}, \mathtt{hi}\}$ *// = the three ranges low, mid, high*

// for one particular composition:
$\mathtt{I}$ *// = all involved instruments*
$\mathsf{T} = (\mathtt{I} \times \mathsf{R}) \nrightarrow \mathbb{N}_1$ *// = all data at one timepoint*
$\mathtt{score} : \mathsf{seq}\, \mathsf{T}$ *// Indexes stand for are all timepoints*
// with at least one event.

// for one particular realization:
$\mathtt{PCStart} : \mathbb{P}\mathsf{C}$ *// = selected pitch classes*
$\mathtt{range} : (\mathtt{I} \times \mathsf{R}) \to (\mathsf{P} \times \mathsf{P})$
$\mathtt{range}(i, r) = (a, b) \implies a \leq b$
// = definitions of the three registers per instrument

// derived types and data:

$$\frac{\mathtt{range}(i, \mathtt{lo}) = (\_, a) \quad \mathtt{range}(i, \mathtt{mid}) = (b, c) \quad \mathtt{range}(i, \mathtt{hi}) = (d, \_)}{\mathtt{gaps}(i) = (\mathtt{key}(b) - \mathtt{key}(a), \mathtt{key}(d) - \mathtt{key}(c))}$$

$$\frac{\mathtt{gaps}(i) = (g, h)}{\mathtt{wellSeparated}(i) \iff g \geq 5 \wedge h \geq 5}$$

$\mathtt{range}' : (\mathtt{I} \times \mathsf{R}) \to \mathbb{P}\mathsf{P}$
$\mathtt{range}'(i, r) = \{\pi_1(\mathtt{range}(i, r)), \ldots, \pi_2(\mathtt{range}(i, r))\}$
$\mathtt{complete}(i) \iff \forall r \in R \bullet \mathtt{PCStart} \subseteq \pi_2 (\!| \mathtt{range}'(i, r) |\!)$

$\mathtt{choose}_{\mathrm{F/C}} : (\mathbb{P}\mathsf{P} \times \mathbb{N}) \to \mathbb{P}\mathsf{P}$

$$\frac{\mathtt{random}(A) = a \quad n > 0}{\begin{array}{l}\mathtt{choose}_{\mathrm{F}}(A, n) = \{a\} \cup \mathtt{choose}(A \setminus \{a\}, n-1) \\ \mathtt{choose}_{\mathrm{C}}(A, n) = \{a\} \\ \qquad \cup\, \mathtt{choose}(A \setminus (\mathsf{O} \times \{\pi_2(a)\}), n-1)\end{array}}$$

$\mathtt{choose}_{\_}(\_, 0) = \varnothing$
$\mathtt{choose}_{\_}(\varnothing, n) = \varnothing$
// = here an error should be reported, see text.

Table 5. Basic definitions to evaluate the "Projection_n" scores.

"4 = middle c". C are all pitch classes, PCStart those selected for the current interpretation. We define C by "zero stands for c" and each octave register extends from an instance of c upward. [23]

The score data type represents the syntactic form of the scores written by Feldman: It is a sequence, where the index stands for all those timepoints which do carry an event. Each value is of type $\mathsf{T} = (\mathtt{I} \times \mathsf{R}) \nrightarrow \mathbb{N}_1$ and maps instruments and registers to the required number of pitches to play. (With most instruments this is implicitly $= 1$, but the piano voices have larger numbers, and so has the solo violin in no. 4.)

The central issue when restricting pitch class repetition is the conflict between the still allowed pitch classes in the course of the execution, according to the selected overall strategy (represented by the set pool), and the ranges for the three registers per instrument (range). [24] The ancillary data $\mathtt{range}'$ contains the defined ranges as sets of pitches;

[23] Be aware that both facts appear natural to the contemporary reader, but are in no way necessary. [17, p. 23], according to [6, p. 327]

[24] For range, gaps, and wellSeparated see above; for pool below.

$\mathtt{eval}_{\mathrm{Free/Instr/Impr/Comp}} : \mathsf{seq}\, \mathsf{T} \to \mathsf{seq}(\mathtt{I} \leftrightarrow \mathsf{P})$

$\mathsf{T}' = (\mathtt{I} \times \mathsf{R}) \to \mathbb{N}$
$\mathtt{score}' : \mathsf{seq}\, \mathsf{T}'$
$\mathtt{score}' = ((\mathsf{dom}\, \mathtt{score} \times \mathtt{I} \times \mathsf{R}) \times \{0\}) \oplus \mathtt{score}$

$\mathtt{pool} : (\mathtt{I} \times \mathsf{R} \times \mathbb{P}\mathsf{C}) \to \mathbb{P}\mathsf{P}$
$\mathtt{pool}(i, r, C) = \mathtt{range}'(i, r) \cap (\mathsf{O} \times C)$
// = pitches currently available for playing

$\boxed{\mathtt{eval}_{\mathrm{Free}}(\mathtt{score})} = \mathtt{evF}(\!| \mathtt{score}' |\!)$
$\mathtt{evF} : \mathsf{T}' \to (\mathtt{I} \leftrightarrow \mathsf{P})$
$\mathtt{evF}(T) = \bigcup i \in I$
$\quad \bullet \{i\} \times \bigcup r \in R \bullet \mathtt{choose}_{\mathrm{F}}(\mathtt{range}'(i, r), T(i, r))$

$\boxed{\mathtt{eval}_{\mathrm{Instr}}(\mathtt{score})} = \bigcup i \in \mathtt{I}$
$\quad \bullet\, \mathtt{eval}_{\mathrm{M}}(\mathtt{PCStart}, \mathtt{extract}(i))$
$\boxed{\mathtt{eval}_{\mathrm{Comp}}(\mathtt{score})} = \mathtt{eval}_{\mathrm{M}}(\mathtt{PCStart}, \mathtt{score})$

$\mathtt{extract}(i) = \lambda(n \mapsto x) \bullet n \mapsto x \cap (\{i\} \times \mathsf{R} \times \mathbb{N}) \;(\!| \mathtt{score} |\!)$
$\mathtt{eval}_{\mathrm{M/J}} : (\mathbb{P}\mathsf{C} \times \mathsf{seq}\, \mathsf{T}) \to \mathsf{seq}(\mathtt{I} \leftrightarrow \mathsf{P})$
$\mathtt{evC} : (\mathbb{P}\mathsf{C} \times \mathsf{T}) \to (\mathbb{P}\mathsf{C} \times (\mathtt{I} \leftrightarrow \mathsf{P}))$

$$\frac{\mathtt{evC}(C, x) = (C', R')}{\boxed{\mathtt{eval}_{\mathrm{M}}(C, x \blacktriangleright \alpha)} = R' \blacktriangleright \mathtt{eval}_{\mathrm{M}}(C', \alpha)}$$

$$\frac{\begin{array}{c} p = \mathtt{pool}(i, r, C) \quad c = \pi_2 (\!| p |\!) \\ n_1 = \# c \quad n_2 < \# c \quad n_3 > \# c \\ p_2 = \mathtt{choose}_{\mathrm{C}}(p, n_2) \\ p_3 = \mathtt{choose}_{\mathrm{C}}(\mathtt{pool}(i, r, \mathtt{PCStart} \setminus c), n_3 - \# c) \\ \mathtt{takeAll} = \{x : c \bullet \mathtt{choose}_F(\mathtt{pool}(i, r, \{x\}), 1)\} \\ \mathtt{evC}(C', \alpha') = (C'', R'') \end{array}}{\begin{array}{l} \textbf{if } \alpha = \{(i, r) \mapsto n_1\} \cup \alpha' \\ \quad \textbf{then } C' = C \setminus c \\ \quad \wedge \boxed{\mathtt{evC}(C, \alpha)} = (C'', (\{i\} \times \mathtt{takeAll}) \cup R'') \\ \textbf{else if } \alpha = \{(i, r) \mapsto n_2\} \cup \alpha' \\ \quad \textbf{then } C' = C \setminus \pi_2 (\!| p_2 |\!) \\ \quad \wedge \boxed{\mathtt{evC}(C, \alpha)} = (C'', (\{i\} \times p_2) \cup R'') \\ \textbf{else if } \alpha = \{(i, r) \mapsto n_3\} \cup \alpha' \\ \quad \textbf{then } C' = \mathtt{PCStart} \setminus (\pi_2 (\!| p_3 |\!)) \\ \quad \wedge \boxed{\mathtt{evC}(C, \alpha)} = (C'', (\{i\} \times (p_3 \cup \mathtt{takeAll})) \cup R'') \\ \textbf{else } \alpha = \varnothing \;\wedge\; \boxed{\mathtt{evC}(C, \alpha)} = (C, \varnothing) \end{array}}$$

$\boxed{\mathtt{eval}_{\mathrm{Impro}}(\mathtt{score})} = \mathtt{eval}_{\mathrm{J}}(\mathtt{PCStart}, \mathtt{score})$

$$\frac{\begin{array}{c} (\_, P_i) = \mathtt{evC}(C, x \cap (\{i\} \times R \times \mathbb{N})) \\ P = \bigcup_{i \in \mathtt{I}} P_i \\ C_{\mathrm{P}} = \pi_2 (\!| \pi_2 (\!| P |\!) |\!) \\ C' = \textbf{if } C_{\mathrm{P}} \subset C \textbf{ then } C \setminus C_{\mathrm{P}} \textbf{ else } \mathtt{PCStart} \setminus (C_{\mathrm{P}} \setminus C) \end{array}}{\boxed{\mathtt{eval}_{\mathrm{J}}(C, x \blacktriangleright \alpha)} = P \blacktriangleright \mathtt{eval}_{\mathrm{J}}(C', \alpha)}$$

Table 6. The implemented four different ways of evaluating the "Projection_n" scores. Terms defined are framed.

complete() is the fact that every such range contains all selected pitch classes—this is not mandatory!

The $\mathtt{choose}_{\_}(A, n)$ function selects from the given set a given number of elements: $\mathtt{choose}_F$ removes only the selected pitch, but $\mathtt{choose}_C$ all instances of the selected pitch class. [25]

[25] As implemented in Table 6. the code may call the function random(..) redundantly, e.g. to select one element from a one element

Table 6: The result of each variant of eval_(..) is always a sequence of values (at the same timepoints as the input data) of type $\mathsf{T} \times \mathsf{P}$, which map instruments to sets of pitches.

(The transformed data T' and score$'$ are for technical reasons only and replace any undefined coordinate in the score data by an explicit zero.)

pool(..) calculates permitted pitches by combining the range restriction for an instrument and its register with a set of pitch classes.

The ancillary functions evF(..) and evC(..) process all score data at one particular timepoint.

$\mathtt{eval}_{\mathrm{Free}}$ simply steps through the timepoints and applies to them evF(..), the random choice for the required number of pitches, without restrictions.

Whenever the set returned by choose_(..) is smaller than the number requested, this is because the particular range has been erroneously defined when preparing the interpretation. It can be checked in advance for all combinations of score position, instrument, and register that the requested number of pitches to play does not exceed the number of playable pitches:

$$\forall n : \mathbb{N}, i : \mathtt{I}, r : \mathsf{R} \\ \bullet\ \mathtt{score}(n)(i,r) \leq \#\mathtt{pool}(i, r, \mathtt{PCStart})$$

This also applies for the other evaluation modes.

$\mathtt{eval}_{\mathrm{Instr}}$ and $\mathtt{eval}_{\mathrm{Comp}}$ both use the ancillary function $\mathtt{eval}_{\mathrm{M}}$, which simply maps evC over the timepoints of the score: The resulting pc set from one timepoint is the input for the processing of its successor.

With $\mathtt{eval}_{\mathrm{Comp}}$ this is done with the complete score at once; with $\mathtt{eval}_{\mathrm{Instr}}$ for each instrument with a filtered score extract(i) separately, merging the results. The function evC(..) realizes the repetition restriction at one particular timepoint: Its input are the score data of type $\mathsf{T} = (\mathtt{I} \times \mathsf{R}) \nrightarrow \mathbb{N}$ and the set of currently allowed pitch classes $\mathbb{P}\mathsf{C}$. It delivers an assignment of pitches $\mathtt{I} \times \mathsf{P}$ and the set of allowed pitch classes for the subsequent timepoint.

Its implementation is *heuristic*: The resource assignment problem as such is NP-hard, but a heuristic approach seems more appropriate and better comprehensible:

First, all combinations of events and registers for which the number of still available pitch classes is *identical* to the number of required pitches ($n_1 = \# c$; the projection π_2 maps a pitch to its pitch class) get the events they need. This is because these assignments must be done in *any* solution which does not start a new pitch class cycle. After the assignment, the function is called again with input data cut down to α' and the set of available pitch classes readjusted to C'.

Please note that the ancillary function takeAll does call the choose operation separately for each pitch class. Calling choose directly for all pitch classes would deliver the same result, but maybe make superfluous calls to random: For pool(..) $= \{$C3, C4, D3$\}$ only one decision is required, namely between the two representatives of pitch class c, but the first of direct random choices could deliver the D3.

If no such event is found, a pair with some degree of freedom and without the need of starting a new cycle is processed ($n_2 < \# c$). This is the point where the algorithm gets heuristic, because collecting all such events and distributing the available pitch classes among them with more global knowledge could bring better results. But in our implementation the assignment p_2 is selected locally. All events of this kind are processed in second line to get as many assignments as possible into the current repetition cycle.

In both cases the input for the recursive call to this function is cut down to the remaining events at this timepoint α' and the remaining pitch classes C'.

With least priority the pairs are treated which require a new cycle anyhow ($n_3 > \# c$): First takeAll uses up the rest of the old cycle. The remaining requests p_3 are chosen from PCStart, amd their classes are subtracted for the starting set C' for the next assignment.

$\mathtt{eval}_{\mathrm{Impro}}$ is different, because it must reflect the principle that the improvising players do not know the decisions of the others before they have heard them. The function steps through the timepoints by the ancillary function $\mathtt{eval}_{\mathrm{J}}$. For each timepoint evC is applied to all voices *separately*, with the same input pc set C. When afterwards all assigned pitches fit into this set (P_i are the assignments calculated for the singleton set $\{i\}$; the inner π_2 maps the pairs from $\mathtt{I} \leftrightarrow \mathsf{P}$ to the pitches, the outer maps these to their pitch classes) then the rest set is taken as the input for processing the subsequent timepoint.[26] Otherwise at least one of the "improvising" instruments has encountered the necessity to start a new cycle. Consequently a new cycle is started also for the whole ensemble: The complete selected set PCStart is the input for the next timepoint, after removing all pitch classes which have been assigned here but are not provided by C.

The software can be downloaded from http://bandm.eu/feldmanProjections.html. It is in the public domain; source code publication is in preparation.

7. AUTOMATED ANALYSES OF THE MODELS

As mentioned in the previous sections, we applied programmed analysis to the data model for sorting out critical properties like overlapping piano chords or string multistops.

The idea of computer based analysis of a work of Feldman has been realized by Thomas Hummel [10]. With an Atari home computer of the Nineties in the programming language Forth, statistical data mostly about pitches and pitch classes are extracted from "Untitled Composition" (1981). However, the fundamental theses and final conclusions are taken "manually".

More recently, David Cline applied statistical analyses to the "graph music of Morton Feldman", i.e. to Projection 1–5 and beyond. His results w.r.t. the "even-handed distribution" to the three registers and to the instruments

set. Any implementation should react appropriately, e.g. not protocol or analyze such a "non-random" request.

[26] Thus accidental doublings in the resulting sum are not the *unavoidable* repetitions from the cycle rule stated above.

```
                  events per register: numbers (percentage):        summed duration per register (idem):
Proj 1        harm  3;10; 7 (= 15.0; 50.0; 35.0 %)   3; 11; 11 (= 12.0; 44.0; 44.0 %)
Proj 1        pizz 18; 7;15 (= 45.0; 17.5; 37.5 %)  18;  7; 15 (= 45.0; 17.5; 37.5 %)
Proj 1        arco  8; 4; 3 (= 53.3; 26.7; 20.0 %)   8;  7;  6 (= 38.1; 33.3; 28.6 %)
Proj 1         sum 29;21;25 (= 38.7; 28.0; 33.3 %)  29; 25; 32 (= 33.7; 29.1; 37.2 %)
Proj 1           harm  20 (26.7 %)   25 (29.1 %) // = events (percentage) per voice, summed-up durations (idem)
                 pizz  40 (53.3 %)   40 (46.5 %)
                 arco  15 (20.0 %)   21 (24.4 %)
Proj 2          fl 16; 6;10 (= 50.0; 18.8; 31.3 %)  48; 10; 21 (= 60.8; 12.7; 26.6 %)
Proj 2          tr 10; 6;17 (= 30.3; 18.2; 51.5 %)  20; 17; 36 (= 27.4; 23.3; 49.3 %)
Proj 2          vl  7;10;10 (= 25.9; 37.0; 37.0 %)  11; 16; 24 (= 21.6; 31.4; 47.1 %)
Proj 2          vc 13;15;23 (= 25.5; 29.4; 45.1 %)  22; 26; 54 (= 21.6; 25.5; 52.9 %)
Proj 2       p1+p2 13; 7;12 (= 40.6; 21.9; 37.5 %)  38; 10; 31 (= 48.1; 12.7; 39.2 %)
Proj 2         sum 59;44;72 (= 33.7; 25.1; 41.1 %) 139; 79;166 (= 36.2; 20.6; 43.2 %)
Proj 2             fl  32 (18.3 %)   79 (20.6 %)
                   tr  33 (18.9 %)   73 (19.0 %)    approximately equal distribution between and intern of
                   vl  27 (15.4 %)   51 (13.3 %)    two subgroups (fl, trp, vl / vc, p) [3, p. 145]
                   vc  51 (29.1 %)  102 (26.6 %)
                p1+p2  32 (18.3 %)   79 (20.6 %)
Proj 3     p1+p1b  1; 6;14 (=  4.8; 28.6; 66.7 %)   1; 14; 41 (=  1.8; 25.0; 73.2 %)
Proj 3     p2+p2b  7; 8; 3 (= 38.9; 44.4; 16.7 %)  17; 19;  3 (= 43.6; 48.7;  7.7 %)
Proj 3        sum  8;14;17 (= 20.5; 35.9; 43.6 %)  18; 33; 44 (= 18.9; 34.7; 46.3 %)
Proj 3        p1+p1b  21 (53.8 %)   56 (58.9 %)
              p2+p2b  18 (46.2 %)   39 (41.1 %)
Proj 4         vlH  6;12;14 (= 18.8; 37.5; 43.8 %)   8; 17; 23 (= 16.7; 35.4; 47.9 %)
Proj 4         vlP 16;10;18 (= 36.4; 22.7; 40.9 %)  16; 10; 18 (= 36.4; 22.7; 40.9 %)
Proj 4         vlA  8;14; 3 (= 32.0; 56.0; 12.0 %)  14; 28;  5 (= 29.8; 59.6; 10.6 %)
Proj 4       p+p2 30;26;30 (= 34.9; 30.2; 34.9 %)  34; 41; 36 (= 30.6; 36.9; 32.4 %)
Proj 4        sum 60;62;65 (= 32.1; 33.2; 34.8 %)  72; 96; 82 (= 28.8; 38.4; 32.8 %)
Proj 4            vlH  32 (17.1 %)   48 (19.2 %)
                  vlP  44 (23.5 %)   44 (17.6 %)
                  vlA  25 (13.4 %)   47 (18.8 %)    "even-handed distribution" [3, p. 140]
                 p+p2  86 (46.0 %)  111 (44.4 %)

...
Projection 4, events by duration:
dura =  1  count =131  (74.4 %)  [1, 2, 5, ...
dura =  2  count = 15  ( 8.5 %)  [224, 193, 290, 68, 101, 293, 137, 202, 205, 17, 274, 149, ...
dura =  3  count = 11  ( 6.3 %)  [256, 288, 36, 132, 170, 267, 299, 141, 13, 207, 239]
dura =  4  count =  4  ( 2.3 %)  [67, 281, 73, 74]
dura =  5  count =  1  ( 0.6 %)  [324]     "a duration he had not yet used" [1, p. 255]
...
Projection 1 --- recognized regions:
sync instruments :
sounding instruments, exact :
thickness 0  length 1: [9, 12, 31, 37, 50, 53, 55, 67, 77, 86, 92, 106, 108, 110, 152, 178]
thickness 0  length 2: [14, 25, 28, 34, 39, 47, 57, 63, 69, 89, 98, 112, 118, 132, 155, 168, ...
thickness 0  length 3: [0, 4, 17, 43, 72, 94, 102, 121, 135, 148, 158, 181]
thickness 0  length 4: [81, 125, 140, 162]
thickness 0  length 11: [188]       "the work's most striking aspect is the longest silence" [3, p. 18]
...
Projection 2 --- recognized regions:
sync instruments :
    2:  [10, 40, 69, 83, 85, 218, 224, 227, 277, 280, 309, 322, 326]
    3:  [11, 27, 58, 61, 79, 228]
    4:  [35, 91, 96, 103, 204]
    5:  [171]
sync keys cross instruments :
    2:  [10, 40, 69, 83, 85, 218, 224, 227, 277, 280, 309, 322]
    3:  [58, 61, 79, 228]
    4:  [35, 91, 96, 103, 204]
    5:  [27, 171]
    9:  [11]
   13:  [326]
sounding instruments, exact :
thickness 0  length 1: [19, 59, 78, 84, 86, 90, 95, 145, 147, 151, 158, 160, 162, 188, 194, 255, ...
thickness 0  length 2: [0, 28, 43, 92, 140, 166, 169, 199, 202, 230, 275, 278, 283]
thickness 0  length 3: [5, 32, 80, 119, 153, 190, 296]
thickness 0  length 4: [21, 46, 53, 65, 99, 213, 244]
thickness 0  length 5: [72, 205, 290]
thickness 0  length 6: [303]
thickness 0  length 9: [104]
thickness 0  length 12: [175]       "the longest [pause] occurs just after the midpoint." [3, p. 19]
thickness 1  length 1: [8, 20, 45, 57, 60, 64, 77, 94, 113, 146, 152, 159, 161, 168, 187, 189, 193, ...
thickness 1  length 2: [2, 12, 25, 30, 38, 41, 70, 156, 225, 263]
thickness 1  length 3: [50, 87, 142, 148, 163, 258, 266, 270, 287]
thickness 1  length 4: [15, 195, 299]
thickness 1  length 7: [237, 248]
thickness 1  length 18: [122]
thickness 2  length 1: [4, 14, 40, 69, 83, 85, 114, 211, 222, 227, 236, 273, 277, 321, 324]
...
thickness 5  length 1: [171, 174, 220, 336]
thickness 5  length 3: [332]        "In the last few seconds, all the instruments play simultaneously." [3, p. 20]
```

Table 7. Parts of the statistical data extracted from the `tscore` model, and its relations to statements by Cline [3] and Vigil [1]. (All start-points given as zero-based "ictus" = quarters.)

```
Projection 2 -- sounding instruments, minimally :
thickness 1  length 1: [20, 45, 77, 79, 83, 85, 91, 94, 103, 146, 152, 159, 161, 168, 187, ...
thickness 1  length 2: [30, 57, 156]
thickness 1  length 3: [2, 25, 50, 69, 87, 96, 142, 148, 163, 210, 266, 280, 287]
thickness 1  length 4: [171, 195, 299]
thickness 1  length 5: [60, 270]
thickness 1  length 6: [113]
thickness 1  length 7: [248, 258]
thickness 1  length 8: [35]
thickness 1  length 11: [8]
thickness 1  length 12: [232]
thickness 1  length 13: [217]
thickness 1  length 18: [122]
thickness 1  length 28: [309]
...
Projection 4 --- recognized regions:
sync instruments :
    2:  [9, 22, 83, 87, 98, 108, 111, 129, 132, 141, 149, 164, 170, 193, 207, 211, 222, 251, ...
sync keys cross instruments :
    2:  [83, 108, 132, 164, 261]
    3:  [22, 111, 170, 207, 222]
    4:  [9, 141, 267]
    5:  [87, 149, 211, 251]
    8:  [129, 193]
    9:  [98]
   11:  [324]
```

"The most impressive [staggered overlapping sequence is] at the very end of the work." [3, p. 19]

"an extremely dense eleven note sonority." [1, p. 255]

Table 8. Statistical data, continued.

in Projection 4 [3, p. 140] are easily reproduced with the `tscore` data models, see Table 7. But is also shows that both properties do not appear in the other pieces.

Once these analyses have been implemented, it took few more lines of code to apply them to the sums of durations. Also here no further equilibria appear.

Speaking of "the longest silence" or "the most impressive [. . .] of staggered sequences" (p. 10, 19, etc.) transits from mere statistical to structural claims; the underlying facts are verified by our analysis.

Cline posits repeated and mirrored "modules" in the "register contours" of the score (p. 164) and "trajectories" (mere graphical phenomena because they transit the limits of the staves) in survived sketches (p. 169). Both are a possible subject of future automated analysis by pattern recognition.

Vigil proposes for an analysis of Projection 4 the notions "restriction", "exclusion", "diversity", "saturation", "density", and "novelty" [1]. It is not quite clear in how far these stand for structural/syntactic properties of the score or for psychological categories in reception. The former could be extracted automatically from the `tscore` model, after appropriate formalization, which appears as promising future work.

8. REANIMATING THE NOTATION

For further research, especially on the psychology of reading and playing by human actors, the first author composed small studies using Feldman's notation ("Drei kleine Studien nach Morton Feldman", Lepper op. el. 20).

Automated performance already showed that with additional restrictions on permitted pitch classes and repetition avoidance, even opposed styles of intention and effect can be supplied. They range from abstract serialism (with precise quasi-mathematical messages to the listener) to easy-listening soundscapes.

It turned out for the first and last of the three movements, that it was appropriate w.r.t. their content to *generate the Feldman scores by an algorithm*. This was remarkably easy (300+200 loc) and proved again the versatility of the basic design of `tscore`.

For the first movement even *score stacking* was applied: A second `tscore` format describes only the curves of the limits of the random values which control the generation of the Feldman events. This approach showed much easier handling of input data, with aesthetically satisfactory outcome.

9. COMPREHENSION

The compositions *Projection 1* to *Projection 5* by Morton Feldman from 1950/51 are important historic avant-garde compositions. They are among the first pieces which founded modern graphic notation and gave the decision about pitches to the players.

`Tscore` is a meta-meta-model model which allows to construct (in few lines of code) a meta-model which allows to create (in little editing time) computer models of all five scores very conveniently. All code is in the public domain—open source publishing is in work.

These score models have been used to reconstruct the concrete graphical appearance, to generate possible sound realizations, and to answer particular analytical questions. During the reconstruction processes of sound and graphics, difficulties became obvious which have not yet been discussed since the days of their composition.

Acknowledgments

Many thanks to Andrew Yoon, NYC, principal developer of Neoscore [15], and to the DigiScore ERC project which supported it. Yoon's work on *Projection 2* inspired our research. Thanks to the anonymous reviewers who helped to improve this article considerably.

10. REFERENCES

[1] R. Vigil, "Compositional Parameters: 'Projection 4' and an Analytical Methodology for Morton Feldman's Graphic Works," *Perspectives of New Music*, vol. 47, no. 1, pp. 233–267, 2009.

[2] A. Noble, *Composing Ambiguity: The Early Music of Morton Feldman*. Farnham, UK: Ashgate, 2013.

[3] D. Cline, *The Graph Music of Morton Feldman*. Cambridge, UK: Cambridge University Press, 2016.

[4] M. Feldman, *Give My Regards to Eighth Street – Collected Writings*. Cambridge, MA: Exact Change, 2000.

[5] B. Boutwell, "Morton Feldman's graphic notation: projections and trajectories," *Journal of the Society for American Music*, vol. 6, no. 4, pp. 457–82, 2012.

[6] M. Lepper, "De linguis musicam notare," Ph.D. dissertation, Universität Osnabrück, Osnabrück, 2021, https://www.epos.uni-osnabrueck.de/buch.html?id=150[20230920].

[7] M. Lepper and B. Trancón y Widemann, "Beam it up! – a classification grid for historic and contemporary practices of beaming by mathematical re-modelling," in *Proceedings of TENOR 2023 conference*, 2023, https://zenodo.org/record/8079306.

[8] F. O'Hara, "New directions in music: Morton Feldman," 1959, (in [4]).

[9] M. Erdmann, "Zusammenhang und Losigkeit," *Musik-Konzepte*, vol. 48/49, 1986.

[10] T. Hummel, *Morton Feldmans "Untitled Composition" (1981) für Cello und Klavier. Eine computergestützte Analyse*. Saarbrücken: Pfau, 1994.

[11] M. Lepper and B. Trancón y Widemann, "tscore: Makes computers and humans talk about time," in *Proc. KEOD 2013, 5th Intl. Conf. on Knowledge Engineering and Ontology Development*, J. Filipe and J. Dietz, Eds., instincc. Portugal: scitePress, 2013, pp. 176–183, http://markuslepper.eu/papers/tscore2013.pdf[20231003].

[12] ——, *Example Instances of the TScore Project Infrastructure*, 2013, http://markuslepper.eu/sempart/tscoreInstances.html[20230920].

[13] ——, "Translets – parsing diagnosis in small dsls, with permutation combinator and epsilon productions," in *Tagungsband des 35ten Jahrestreffens der GI-Fachgruppe "Programmiersprachen und Rechenkonzepte*, ser. IFI Reports, vol. 482. University of Oslo, 2018, pp. 114–129, http://urn.nb.no/URN:NBN:no-65294[20230920].

[14] A. Danial, "cloc – count lines of code (v1.92)," Tech. Rep., 2021, https://github.com/AlDanial/cloc [20231003]. [Online]. Available: https://doi.org/10.5281/zenodo.5760077

[15] A. Yoon, "Neoscore – notation without bars," Tech. Rep., https://github.com/DigiScore/neoscore/tree/main/examples/feldman_projection_2 [20230920].

[16] O. Wiener, *Morton Feldmans "The Viola in my Life" (1970–71)*. Saarbrücken: Pfau, 1996.

[17] H. Riemann, *Notenschrift und Notendruck. Festschrift zur 50-jährigen Jubelfeier des Bestehens der Firma Röder*. Leipzig: C.G. Röder GmbH, 1896.

[18] J. Spivey, *The Z Notation: a reference manual*. Prentice Hall, 1988, https://github.com/Spivoxity/zrm/blob/master/zrm-pub.pdf[20231003].

A. MATHEMATICAL NOTATION

The employed mathematical notation is fairly standard, inspired by the Z notation [18]. The following table lists some details:

$\mathbb{N}$	All natural numbers, incl. Zero.
$\mathbb{N}_1$	All natural numbers without Zero.
$\#A$	The cardinality of a finite set = the natural number of the elements contained.
$\mathbb{P}A$	Power set, the type of all subsets of the set A, incl. infinites.
$A \setminus B$	The set containing all elements of A which are not in B.
$A \times B$	The product type of two sets A and B, i.e. all pairs $\{c = (a, b) \mid a \in A \land b \in B\}$.
π_n	The nth component of a tuple.
$A \rightarrow B$	The type of the *total* functions from A to B.
$A \nrightarrow B$	The type of the *partial* functions from A to B.
$A \leftrightarrow B$	The type of the relations between A and B.
$a \mapsto b$	An element of a relation; simply another way to write (a, b)
$f (\!\mid s \mid\!)$	The image of the set s under the function or relation f
$\mathsf{dom}\,A, \mathsf{ran}\,A$	Domain and range of a relation.
$r \oplus s$	Overriding of function or relation r by s. Pairs from r are shadowed by pairs from s: $r \oplus s = \big(r \setminus (\mathsf{dom}\,s \times \mathsf{ran}\,r)\big) \cup s$, with dom and ran being domain and range, resp.
$\mathsf{seq}\,A$	The type of finite sequences from elements of A, i.e. of maps $\mathbb{N} \nrightarrow A$ with a contiguous range $\{1..n\}$ as its domain. Instances are notated by listing the range elements in $\langle \ldots \rangle$.
$a \blacktriangleright \beta$	A sequence seen as a first element a and the rest sequence β (our extension). Same as $\langle a \rangle$ concatenated with β.

The frequently used notation

$$\begin{array}{c} a \\ b \qquad c \\ \hline d \end{array}$$

means as usual $a \land b \land c \implies d$. Nearly always it should be read as an *algorithm*: "For to calculate d, try to calculate a, b and c. If this succeeds, the answer d is valid."

Functions are considered as special relations; relations as sets of pairs. So with functions, expressions like "$f \cup g$" are defined.