F-OPERATOR:
A PROPOSAL FOR A GENERATIVE NOTATION FOR FREQUENCY FACTORS IN MUSIC

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ABSTRACT

In music theory and historical musicology, tuning systems can be discussed from a variety of angles, ranging from historically given contingent facts or hypotheses, over physiological and psychological experiments, up to integration into an aesthetic theory. All these different approaches define specific collections of objects and calculation rules and live on a symbolic side.

Eventually, these calculations deliver numeric factors to be applied to some reference frequency. These form the factor side. We propose F-operator, a notation system for these values which allows easy identification and labeling of commonly used factors and thus easy comparison of distinct systems from the symbolic side. F-operator is equally well suited for writing, talking, speech recognition, sign language, Braille, and computer representation.

1. PROBLEM ANALYSIS

1.1 Separation of two sides

In music theory and historical musicology, tuning systems can be discussed from a variety of angles, ranging from historically given contingent facts or hypotheses, over physiological and psychological experiments, up to integration into an aesthetic theory. A wide diversity of approaches can also be found in harmonic theories, in physiological research, and in many other realms exploring the perception of music.

All these disciplines can have more or less precise and very different "algebras" (in the widest sense of the word) as their modelling means. The elements of these algebras shall be called *pitches* in the following; all these algebras live on a *symbolic side*.

But eventually all symbolic systems deliver some *factors*. This corresponds to the human way of reception: A particular factor between any two audible frequencies is received as the same interval; a particular sequence of factors between adjacent tones is received as the same melody, independent from the absolute starting point. [1, p. 10] [1] The frequencies and factors live on the *factor side*.

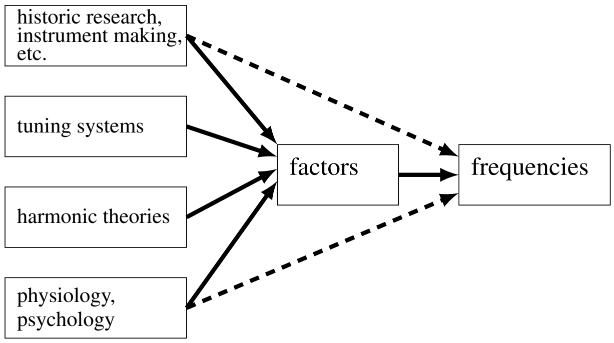

Figure 1. Research realms and their relations to factors and frequencies.

See Figure 1 for a simplified depiction of the situation: Most realms indeed talk (mainly) about the factors between frequencies, [2] only few directly about frequencies: historic research on tuning system does, physiology when addressing duration values of nerve reaction times, etc. [3] [4]

All the more, a concise naming system is helpful. Here we propose the F-operator family, a notation system for the factors on the frequency side, which (a) reflects a minimal and canonical way of their generation, and so (b) supports comparison of and talking about the different systems on the symbolic side.

1.2 The naming problem

The fundamental problem in giving names to frequency factors is that all existing approaches operate on the very

[1] Remarkably, we could not find an earlier scientific publication of this fundamental fact. Even Helmholtz [2] does not mention it; it seems it has been mere folklore ever since.

[2] In times when the physical concepts of oscillation and frequency were not yet known, these factors were given as ratios of string lengths.

[3] And esoterics do: Renold [3] quotes Rudolf Steiner "c[=C3] = 128 Hz = Sonne" [p.108] "[ist] heute der einzige für den westlichen Menschen wirklich geeignete Kammerton." ("is today for Western people the only appropriate tuning.") [p. 13].

[4] In most use cases, also the values on the frequency side exist as *intentions*, as *concepts*. Euler already recognized the twelve-tone equal temperament tuning as an auxiliary means which can be perceived by a listener as representing a just intonation ("Substitutionstheorie" / "theory of substitution" [4, p. 163 ff] [5, p. 125]; see also [6, p. 49]) and contemporary mathematical approaches see those frequencies as the center of fuzzy ranges [7, p. 60].

F 1/2	=	uF 1/2	=	$\sqrt{2}$	=	1.4142...	=	600	cent
		uFn 6	=	$2^{-9} * 3^{6}$	=	1.4238..	=	611.7300...	cent
		uFn 2 -2	=	$2^{2} * 3^{2} * 5^{-2}$	=	1.44	=	631.2826...	cent
Fn 0 -1 1	=	uFn 0 -1 1	=	7/5	=	1.4	=	582.5122...	cent
uF 11↑1	=	uFn 0 0 0 1	=	11/8	=	1.375	=	651.3179...	cent

Table 1. Different intervals which all are sometimes called "tritone".

different symbolic sides to find names for the general and common factor side. Significant examples are:

English wikipedia, lemma "Semitone", section "Just_intonation" [5] : "Just chromatic semitone, chromatic semitone, or smaller [chromatic semitone], or minor chromatic semitone [vs.] Larger chromatic semitone or major chromatic semitone, or larger limma, or major chroma, [vs.] Just diatonic semitone or smaller [diatonic semitone], or minor diatonic semitone, [vs.] Larger diatonic semitone or greater [diatonic semitone] or major diatonic semitone," — "Descartes used the word schisma [for] the syntonic comma (81:80)."

Ibid., lemma "Diesis" [6] : "As a comma, the above-mentioned 128:125 ratio is also known as the lesser diesis, enharmonic comma, or augmented comma."

Haluška [7, p. xxix] calls thirteen just fifths modulo octave (in F-operator written as uFn 13 = F -8 13, see below) a "Pythagorean double diminished fifth", and "512/375" (p. xxvii) (= uFn -1 -3) a "double diminished fifth". But all these names are meaningful only in some particular pitch calculus on the symbolic side. Figure 2 shows a doubly diminished fifth, which is an appropriate naming from functional harmonic analysis, describing the downward step from D//9- to D//5+, both pitch roles being phenomena with a complicated historical genesis. [7] This can be realized by a step from cFn -1 -1 to cFn 0 2 of size cFn 1 3.) This is obviously a quite different mental object than the above-mentioned historically neutral frequency factors.

There are prominent examples from the community of micro-tonal composers and researchers:

A text file published by the *Huygens-Fokker Stichting* [8] lists 604 rational numbers, each with a name but without any explanation, up to the 26-digit

$$\frac{19383245667680019896796723}{19342813113834066795298816} .$$

Another proposal by Keenan (1999) [9] is much more systematic, but restricted to rational intervals. His table may be of high value for the particular underlying theory, but not in general. A sentence like "I have included the interval 8:13 [...], because it is certainly a better-known representative of the category of *neutral sixth* than is 11:18" shows both: that a certain degree of arbitrariness is involved, and that the naming is tightly related to one particular theory.

[5] https://en.wikipedia.org/wiki/Semitone#Just_intonation[20251103]

[6] https://en.wikipedia.org/wiki/Diesis[20251103]

[7] We use the computer-compatible funCode notation for functional analysis, because it is the only with precisely defined mathematical semantics: "D" means the triad of the dominant; "D/" the same without its root pitch class; "D//n" cancels all default triad components and refers to the chord component n only. In functional theory, the root of the second chord D/5+79- in Figure 2 is a cancelled b flat. [8]

[8] https://www.huygens-fokker.org/docs/intervals.html [20251222]

[9] https://www.dkeenan.com/Music/IntervalNaming.htm[20251222]

Figure 2. Indeed a doubly diminished fifth in Bruckner, First Symphony, Linz version, mv. 1, m. 102.

1.3 The F-operator approach

The basic idea of F-operator is that a combination of leading alphabetic characters and trailing numbers names the fundamental frequency factors in music theory.

Examples from technical history show that (a) such combinations easily can receive an "auratic identity", which makes them recognizable and easy to speak and remember, like "Z 80", "PDP 11", or "HAL 9000".

Furthermore, (b) the sequence of numbers shall reflect directly the derivation of the factors from the series of partial tones, an idea proposed (earlier but independently) by Joe Monzo [9, 10], while the leading characters establish the fundamental categories of derivation.

The result is a format neutral to the different symbolic sides and very compact to write and speak. For instance, "F0" is the factor 1.0, normally called "perfect prime", and "F1" is the "perfect octave". "uFn 1" it the "just fifth" and "mFn 1" the "just forth". "uFn12" is the "Pythagorean comma" and "F1/12" the half-tone in 12 EDO tuning. [10]

Table 1 shows that F-operator is suited to unambiguously and easily label intervals which are often mixed up.

2. ARCHITECTURE OF F-operator

2.1 Goals and Intentions

The proposal F-operator addresses multiple difficulties in discussing about tuning and harmonic systems, and in many others of the above-listed realms:

[G-1] It allows to use a common notation of the effects on the factor/frequency side, which are independent of all, and common to almost all, generating theories on the symbolic side. This allows (a) a clean separation of these two

[10] 12 EDO means *Equal Division of the Octave by twelve*. We prefer this nomenclature to "12 TET", because "12 tone equal temperament" implies a particular application context, namely to "temper out" some irregularities from some "original" tuning.

sides, and (b) gives a tertium comparationis for the different generating systems on the symbolic side.

[G-2] Most systems on the symbolic side assign own names to their results on the factor side. This contributes to the overwhelming plethora of synonyms found in literature which shall be replaced, see the examples above.

[G-3] The structure of the numeric label shall reflect one possible method for *deriving* the interval—it shall be *generative*. This provides a more characteristic mental model.

[G-4] There is danger of confusion, especially when discussing the relation between the harmonic spectrum and harmonic theories, between the physical intervals relative to the base frequency, or relative to the next-lower octave of the base frequency, or about all frequency factors which only differ by octaves, etc. For all these cases the *multiple* functions defined by F-operator shall improve clarity.

[G-5] All defined functions are well-formed mathematical objects and can immediately be used for further mathematical constructions, see Table 5 for examples.

[G-6] All expressions shall be written, read, and spoken as easy as possible, with minimal formal noise. They are equally well-suited for Braille, sign language, voice recognition, and computer representation.

2.2 Construction

Table 2 shows the complete construction of F-operator. The type $\widehat{\mathbf{F}}$ is its central data structure. Each instance contains *tuples of exponents*, called *f-tuple*, for "frequency tuple" or "factor tuple". The first line of the table defines $\widehat{\mathbf{F}}$ as all total functions from $\mathbb{N}$ into rationals which have only finite support (positions with a value different from zero).

The constructor F takes a finite, possibly empty list of rationals and delivers an instance of $\widehat{\mathbf{F}}$: All positions are set to zero and then overridden by the explicit input.

For lean definitions, the technical types of all structures have been chosen as relations, even when they are functional. Function f evaluates tuples from $\widehat{\mathbf{F}}$ into real numbers: The contained rational numbers are used as exponents for the sequence of prime numbers $\mathcal{P}$ and the results are multiplied. The assignments between bases and exponents are by their respective position in both sequences: the leftmost exponent in $e : \widehat{\mathbf{F}}$ is taken as the exponent of base two, the next that of base three, of five, of seven, etc. The operators $*$, $/$, and $\uparrow$ are implemented pointwise.

All this produces a somehow peculiar algebraic structure: $\widehat{\mathbf{F}}$ is not a complete ring, but only a multiplicative group. It contains only a subset of the algebraic numbers: Since there is no addition, the constant of the golden ratio $\mathsf{F}\ \ {-1} +$ $\mathsf{F}\ \ {-1}\ 0\ 1/2$ is not contained. Nevertheless, the factors represented by $\widehat{\mathbf{F}}$ cover most known tuning systems.

Relation $\widehat{\mathsf{c}}$ maps each real number to all of its products with any power of two. This corresponds to factors which determine frequencies which differ only in their *octave register*. In the symbolic world this kind of abstraction is often called *pitch class*; accordingly, each $\widehat{\mathsf{c}}(X)$ can be called a *frequency class* or *factor class*.

The function c does the same for f-tuples. Table 2 gives two definitions: The second term of the equation tells us to go from the f-tuple to the factor by applying f, then apply $\widehat{\mathsf{c}}$, and finally go back again, as depicted in the diagram.

$$\widehat{\mathbf{F}} = \{f : \mathbb{N} \to \mathbb{Q} \mid f \rhd \{0\} \text{ finite}\}$$
$$\mathsf{F} : \mathsf{SEQ}\,\mathbb{Q} \leftrightarrow \widehat{\mathbf{F}}$$
$$\mathsf{F} = \lambda e : \mathsf{SEQ}\,\mathbb{Q} \bullet (\lambda i : \mathbb{N} \bullet 0) \oplus e$$

$$\mathsf{f} : \widehat{\mathbf{F}} \leftrightarrow \mathbb{R}$$
$$\mathsf{f} = \lambda e : \widehat{\mathbf{F}} \bullet \prod\{i : \mathbb{N} \bullet \mathcal{P}(i)^{e(i)}\}$$

$$(*) : \widehat{\mathbf{F}} \times \widehat{\mathbf{F}} \leftrightarrow \widehat{\mathbf{F}}$$
$$(*) = \lambda e, f : \widehat{\mathbf{F}} \bullet \lambda i : \mathbb{N} \bullet e(i) + f(i)$$

$$(/) : \widehat{\mathbf{F}} \times \widehat{\mathbf{F}} \leftrightarrow \widehat{\mathbf{F}}$$
$$(/) = \lambda e, f : \widehat{\mathbf{F}} \bullet \lambda i : \mathbb{N} \bullet e(i) - f(i)$$

$$(\uparrow) : \widehat{\mathbf{F}} \times \mathbb{Q} \leftrightarrow \widehat{\mathbf{F}}$$
$$(/) = \lambda e, f : \widehat{\mathbf{F}} \bullet \lambda i : \mathbb{N} \bullet e(i) * f$$

$$\widehat{\mathsf{c}} : \mathbb{R} \leftrightarrow \mathbb{R}$$
$$\widehat{\mathsf{c}} = \{x, y : \mathbb{R} \mid \exists z : \mathbb{Z} \bullet x \cdot 2^z = y\}$$

$$\widehat{\mathsf{u}}, \widehat{\mathsf{m}} : \mathbb{R} \leftrightarrow \mathbb{R}$$
$$\widehat{\mathsf{u}} = \lambda x : \mathbb{R} \mid 1 \leq x < 2 \bullet x$$
$$\widehat{\mathsf{m}} = \lambda x : \mathbb{R} \mid 1/\sqrt{2} < x \leq \sqrt{2} \bullet x$$

$$\begin{tikzcd}
\mathsf{SEQ}\,\mathbb{Q} \arrow[r, "\mathsf{F}"] & \widehat{\mathbf{F}} \arrow[r, dotted, "\mathsf{c}"] \arrow[d, "\mathsf{f}"'] & \widehat{\mathbf{F}} \arrow[r, dotted, "\mathsf{u},\mathsf{m}"] \arrow[d, "\mathsf{f}"'] & \widehat{\mathbf{F}} \arrow[d, "\mathsf{f}"] \\
& \mathbb{R} \arrow[r, "\widehat{\mathsf{c}}"'] & \mathbb{R} \arrow[r, "\widehat{\mathsf{u}},\widehat{\mathsf{m}}"'] & \mathbb{R}
\end{tikzcd}$$

$$\mathsf{c}, \mathsf{u}, \mathsf{m} : \widehat{\mathbf{F}} \leftrightarrow \widehat{\mathbf{F}}$$
$$\mathsf{c} = \mathsf{f}^{\sim} \circ \widehat{\mathsf{c}} \circ \mathsf{f} = ((*) \circ \mathsf{F})(\!|\mathbb{Z}^1|\!)$$
$$\mathsf{u} = \mathsf{f}^{\sim} \circ \widehat{\mathsf{u}} \circ \mathsf{f}$$
$$\mathsf{m} = \mathsf{f}^{\sim} \circ \widehat{\mathsf{m}} \circ \mathsf{f}$$

Table 2. Construction of F-operator.

The third term operates directly on f-tuple arguments: Build all singleton sequences of integer numbers, convert these to f-tuples, and multiply them with the arguments.

Both $\widehat{\mathsf{c}}$ and c are frequently found in harmonic theories, when intervals are discussed under abstraction from the octave registers of the concrete sounds.

The functions $\widehat{\mathsf{u}}$ and $\widehat{\mathsf{m}}$ simply filter their arguments and let pass only those factors which falls in the semi-closed range given in the formulas—u and m do the same for f-tuples. Applied to octave classes, they select one factor.

Let $x : \mathbb{R}$ be a frequency factor, then both $\mathsf{u}(\mathsf{c}(x))$ and $\mathsf{m}(\mathsf{c}(x))$ select *a special representative* from the factor's class: It corresponds to the musical interval which is **u**pward and smaller than a perfect octave, or upward or downward of **m**inimal size, respectively.

$$\begin{aligned}
\mathsf{F}\ a\ b\ c &= \mathsf{F}(\!|\{(a,b,c)\}|\!) && = \{\mathsf{F}(a,b,c)\}\\
\mathsf{cF}\ a\ b\ c &= \mathsf{c}(\!|\mathsf{F}\ a\ b\ c|\!) && = (\mathsf{c}\circ\mathsf{F})(\!|\{(a,b,c)\}|\!)\\
\mathsf{uF}\ a\ b\ c &= \mathsf{u}(\!|\mathsf{cF}\ a\ b\ c|\!) && = \{(\mathsf{u}\circ\mathsf{c}\circ\mathsf{F})(a,b,c)\}\\
\mathsf{mF}\ a\ b\ c &= \mathsf{m}(\!|\mathsf{cF}\ a\ b\ c|\!) && = \{(\mathsf{m}\circ\mathsf{c}\circ\mathsf{F})(a,b,c)\}
\end{aligned}$$

$$\begin{aligned}
\mathsf{Fn}\,b\ c &= \mathsf{F}\ 0\ b\ c\\
\mathsf{cFn}\,b\ c &= \mathsf{c}(\mathsf{Fn}\ \ b\ c)\\
\mathsf{uFn}\,b\ c &= \mathsf{u}(\mathsf{cFn}\ \ b\ c)\\
\mathsf{mFn}\,b\ c &= \mathsf{m}(\mathsf{cFn}\ \ b\ c)
\end{aligned}$$

$$\mathsf{F}\ a{\uparrow}b\ \ c{\uparrow}d \quad = \quad \mathsf{F}\ \underbrace{0\,0\ldots0}_{\mathcal{P}^{\sim}(a)}\ b \quad \underbrace{0\,0\ldots0}_{\mathcal{P}^{\sim}(c)-1-\mathcal{P}^{\sim}(a)} \quad d$$

$$\begin{aligned}
\mathsf{u}(r:\mathbb{R}) &= \widehat{\mathsf{u}}(\!|\widehat{\mathsf{c}}(\!|\{r\}|\!)|\!) && = \{(\widehat{\mathsf{u}}\circ\widehat{\mathsf{c}})(r)\}\\
\mathsf{m}(r:\mathbb{R}) &= \widehat{\mathsf{m}}(\!|\widehat{\mathsf{c}}(\!|\{r\}|\!)|\!) && = \{(\widehat{\mathsf{m}}\circ\widehat{\mathsf{c}})(r)\}
\end{aligned}$$

Table 3. Abbreviated writing of F-operator functions.

Thus $\mathsf{u}(\mathsf{c}(\mathsf{F}\,0\ \ 1))$ corresponds to the pure fifth upwards and $\mathsf{m}(\mathsf{c}(\mathsf{F}\,0\ \ 1))$ the pure fourth downwards.

2.3 Shortcuts; Abbreviated Writing and Pronunciation

Additional rules concerning the written and spoken representation of all these functions aim at a compact and specific appearance, see Table 3:

1. Each nesting of function names from F-operator can be written and spelled as collapsed into one single function name.
2. When embedded in prose text, the argument list of all functions can be written and spoken without framing parentheses, and with simple whitespace as separators. (The latter has been inspired by [9].)
3. The F operator binds weaker than the division, so that the simple sequence $\mathsf{uFn}\,1/2\ \ 1/13$ means two rational exponents. More complex expressions must use parentheses, as in $\mathsf{uFn}(1/13{*}r)$ or $(\mathsf{uFn}\,1/13){*}r$.
4. In that abbreviated notation, the very first tuple component can be the predefined constant n. This is sensible to indicate that the octave value is "don't care", because in any case the tuple shall be argument to the c function. This n is also collapsed into the compound function word, which gives the frequently used constructor cFn.
5. When the context requires a number, the function f is applied implicitly to singleton values from $\widehat{\mathbf{F}}$.
6. Because u and m select from classes, the class building operation c is applied implicitly.
7. Higher exponents can be named explicitly instead of by position. For instance $\mathsf{F}\,11{\uparrow}b\ \ 13{\uparrow}d$ allows to speak about powers of the mentioned primes without the need to count a non-trivial prefix of zeros.

The last two lines in Table 3 make u and m directly applicable to factors instead of f-tuples, by overloading. Therefore u 5 and m 3 can be used to construct the pure third and pure forth, respectively.

2.4 Evaluating to Rationals

Restricting the arguments of F to integer values yields the restricted functions

$$\overline{\mathsf{F}}, \overline{\mathsf{uF}}, \overline{\mathsf{mF}} : \mathsf{SEQ}\ \mathbb{Z}\ \rightarrow\ \mathbb{Q}$$

Sequences of integer exponents can thus be evaluated to a rational number. Whenever two such rationals are treated as pitches, by applying them to the same reference pitch and frequency, their arithmetic quotient stands for the interval between those pitches.

Whenever two such rationals stand for two intervals, one often wants to compare their sizes. A usual method is to take the rational values written as fractions, convert them to the least common denominator, and compare the numerators. Then the ratio of the numerators corresponds to the interval between the intended pitches. For instance, it turns out that four natural fifths $= \mathsf{uFn}\,4\ = 81/64$ are larger than one natural third $= \mathsf{uFn}\,0\ 1\ = 5/4 = 80/64$, and that the interval between them, the "syntonic comma", is $81/80$. Written completely in F-operator, it holds that $\mathsf{u}(\mathsf{Fn}\,4\ /\ \mathsf{Fn}\,0\ 1) = \mathsf{uFn}\,4\ {-1}$.

Another method of comparing is to write the rational results as *decimal fractions*. These are nothing else than compound numbers adding an optional integer part to the left of the decimal dot, an optional fractional part with a denominator 10^k, plus an optional periodically infinite fractional part with a denominator $10^m * (10^n - 1)$, with $k, n : \mathbb{N}_1$ and $m : \mathbb{N}$. This way of writing also makes interval sizes obvious. But because of the somehow arbitrary base "10", most figures do no longer indicate the underlying construction. The syntonic comma becomes 1.0125, and its size can be compared to, for instance, the Pythagorean comma $\mathsf{uFn}\,12 = \mathsf{F}\,{-19}\ 12 = 531441/524288 = 1.013643..$

Comparing both intervals in the F-operator notation is done by division: $\mathsf{F}\,{-19}\ 12/\,\mathsf{F}\,{-4}\ \ 4\ \ {-1}\ = \mathsf{F}\,{-15}\ \ 8\ \ 1\ = 32805/32768 = 1.00113..$

2.5 Evaluating to Reals

If and only if there is a non-integer rational among the exponents, the resulting frequency factor is not rational any more. As a consequence, there is no simple number format to notate it, and also in common computer systems most representations are only approximations. While this restriction is almost never relevant in signal-processing practice, it is important to keep it in mind for a correct understanding of the nature of these intervals.

Furthermore, for to get an intuitive feeling for interval sizes, for instance when singing, it is desirable to have in the world of exponents, which determines our perception [1, **?**, p. 10] some unifying mechanism which allows easy and intuitive comparison, as the common denominator does in the world of factors.

Common practice is to convert the exponential representations which involve several different bases to one single exponent of the single basis two. Thus a common measure for interval size is $\mathsf{f}(\mathsf{F}(1/12 * 1/100))$, which is called the *cent interval*. The postfix unit "cent" is append to logarithms of frequency factors. To convert cent values back to factors, an exponential function must be applied. For

	$\log\_ \in \mathbb{Q}$ cent	$\log\_ \notin \mathbb{Q}$ cent
$\mathsf{F}(\_) \in \mathbb{Q}$	$\mathsf{F}\,2 = 4 = 2400\,\mathsf{cent}$ *(double octave)*	$\mathsf{uFn}\,0\;1 = 5/4 = 386.314\ldots\mathsf{cent}$ *(third in just intonation)*
$\mathsf{F}(\_) \notin \mathbb{Q}$	$\mathsf{F}(1/2) = \sqrt{2} = 600\,\mathsf{cent} = 1.41421\ldots$ *(half octave)*	$\mathsf{uFn}(1/13) = \sqrt[13]{3} = 1.0881822\ldots = 146.3042\ldots\mathsf{cent}$ *(Bohlen–Pierce tempered half-tone)* $\mathsf{uF}(1/2)\;2 = \sqrt{2} * 9/8 = 1.5910\ldots = 803.9100\ldots\mathsf{cent}$ *("formed" G♯ from [3], see sect. 3.6)*

Table 4. Examples for combinations of rational and irrational factors and Cent values.

isolated values, this conversion can be left implicit, see Table 4. In practice, often decimal fractions are notated followed by this unit, to approximate real numbers.

2.6 Rational and Irrational Categories of f-tuples

According to the theorem of Lindemann and Weierstraß [11], those cent value of factors $\neq 1$ with other bases than 2 cannot be rational. The conversion to cent values is frequent in practice and indispensable for prima-vista intuition for any reader, for didactic demonstrations, for experiments in psychology and physiology, and many more.

But in harmonic theory it is hardly useful, since it is (nearly always) only an approximation, which does not explain the nature (Hegel: "Wesen") of an interval, but obscures it.

The authors have spent considerable time and effort in correcting the German Wikipedia pages. What frequently has to be done is (a) prepend the text "approximated" to table headers, (b) write the cent values with trailing dots, and/or (c) replace "=" by "≈".

This is tedious, but necessary. Especially laymen and beginners must be protected from the false impression that music intervals are given their characteristics by some "mystical method of quantification". Indeed, it is by comparatively simple *qualitative* derivation.

As a consequence, all f-tuples fall in one of four categories: The factor is rational and the cent value is rational, if and only if there is only one integer exponent, and this is for the base two. (This exponent may be zero.)

The factor is irrational and the cent value is rational, if and only if there is only one non-integer rational exponent and its base is two.

The factor is rational and the cent value is irrational, if and only if only integer exponents are included, and at least one base other than two.

The factor is irrational and the cent value is irrational, if and only if at least one non-integer exponent and one base other than two are included.

Table 4 shows these rules in formulas and typical examples for all cases. Every computer implementation of F-operator must make very clear which kind of numbers is *meant*, independently of the implementing data type of the employed language. In this article we use the format "1.234.." with two dots for (voluntary) abbreviations of rational numbers, but "1.234..." with three dots for (necessary) abbreviations of irrational numbers.

3. EXAMPLE APPLICATIONS

3.1 Deriving the pure fifth from the third partial tone

Nearly all theories on common Western harmonics have the *triad* as their basic material, and derive its structure from that of the spectrum of partial tones, the harmonic series. The frequency factors to be applied to the base frequency are $\mathsf{F}\,1$, $\mathsf{F}\,0\;1$, $\mathsf{F}\,0\;0\;1$, $\mathsf{F}\,0\;0\;0\;1$, etc.

When we define all harmonic effects to happen "modulo octave"—which certainly is a simplification, but a starting point— then $\mathsf{F}\,0\;1$ is the first interval from the list above which generates harmonic content. From this the *just fifth* is derived, as the framing interval of the triad. This derivation can be achieved either by setting the frequency in relation to the next lower power of 2, which is the interval of the octave. So the just fifth can be derived and written as $\mathsf{F}\;{-1}\;1$. Or it can be done by our class construction and picking the smallest representative greater than one by $\mathsf{uFn}\,1$. It holds

$$\mathsf{F}\;{-1}\;1 = \mathsf{uFn}\,1 = 3/2$$

Stacking arbitrary many just fifths, we never reach the frequency class of the starting point, i.e. its frequency modulo octave, again.

$$a \neq b \Longleftrightarrow \mathsf{uF}(0, a) \neq \mathsf{uF}(0, b)$$

No power of 2 (except $2^0 = 1$) is also a power of 3; so we get the well-known "spiral of fifths". This constitutes a scale, unlimited to both sides, called *Pythagorean*, as

$$\{z : \mathbb{Z} \bullet \mathsf{uF}(\_, z)\}$$

Typical scales for music purposes are contiguous segments from this scale, typically of five to seven steps, see for instance most of the Japanese scales in [12]. A few others described there contain gaps and thus semitones, see "Hirajoshi" and followers in Table 10.

Table 6 shows the first powers of 3, sorted by increasing exponents. After twelve fifths we reach a frequency factor which has a significantly smaller distance to its lower neighbor (in the scale constructed so far) than its predecessors. It thus has a different sound quality and cannot be used for melodic or harmonic purposes in the same way as all the others. This interval is the *Pythagorean comma* $= \mathsf{uFn}\,12 = \mathsf{F}\;{-7}\;12 = 531441/524288 = 1.0136433 \approx 23.46\,\mathsf{cent}$.

$$\mathsf{arrangedPureMajorTriads} = \{R \subset \mathsf{cF}(0); T \subset \mathsf{cF}(0,0,1); F \subset \mathsf{cF}(0,1) \mid R \neq \varnothing \wedge T \neq \varnothing \wedge F \neq \varnothing \bullet R \cup T \cup F\}$$
$$\mathsf{arrangedEDOMajorTriads} = \{R \subset \mathsf{cF}(0); T \subset \mathsf{cF}(1/3); F \subset \mathsf{cF}(7/12) \mid R \neq \varnothing \wedge T \neq \varnothing \wedge F \neq \varnothing \bullet R \cup T \cup F\}$$

Table 5. Use of F-operator expressions directly in mathematical formulas.

uF 0	=	fF 0	=	1	=	1	=	1	=	0 cent
uFn 1	=	fF −1 1	=	3/2	=	1.5	=	1.5	≈	701.9550 cent
uFn 2	=	fF −3 2	=	9/8	=	1.125	=	1.125	≈	203.9100 cent
uFn 3	=	fF −4 3	=	27/16	=	1.6875	=	1.6875	≈	905.8650 cent
uFn 4	=	fF −6 4	=	81/64	=	1.265625	=	1.265625	≈	407.8200 cent
uFn 5	=	fF −7 5	=	243/128	=	1.8984375	=	1.898438..	≈	1109.7750 cent
...										
uFn 11	=	fF −17 11	=	177147/131072	=	1.35152435302734375	=	1.351524..	≈	521.5050 cent
uFn 12	=	fF −19 12	=	531441/524288	=	1.0136432647705078125	=	1.013643..	≈	23.4600 cent
uFn 13	=	fF −20 13	=	1594323/1048576	=	1.52046489715576171875	=	1.520465..	≈	725.4150 cent
etc.										

Table 6. Examples of usage: The Pythagorean scale and the Pythagorean comma.

The next twelve steps will thus be simply the first twelve, shifted by this “non-interval”. Therefore most constructions of Pythagorean scales end here.

3.2 Find multiple pure fifths which come near an octave

Two sequences of two different iterated intervals (here: octave and pure fifths) which only meet once, come arbitrarily near elsewhere.

This lemma together with the Pythagorean comma found above, has since long been a motivation to find further near-misses of the spiral of fifths and the octaves. uF 53 motivated the Mercator–Tanaka division of the octave by cF 1/53, see section 6.2 below. The cycle up to cFn 360 had already been found in fourth century China and fitted well into Chinese philosophy: “Including the old tones, there are a total of 360, one for each day.” [13, p. 216].

3.3 Intervals derived from higher components of the harmonic series

Nearly all harmonic theories employ in their calculus further intervals derived from the harmonic series. A systematic way to enumerate candidates is to take descending pairs. In most existing theories, their higher value is subject to a particular limit.

The next interesting element is the fifth one = F 0 0 1. Its next lower octave and its immediate predecessor is F 2, so that uFn 0 1 = F -2 0 1 = $5/4$. This is commonly used to derive the interval *just major third*.

The ratio to the next predecessor is F 0 -1 1 = $5/3$, is a major sixth. The minimal representative mF 0 -1 1 = $5/6$ is less than one, the interval goes *downward*. Swapping the two compared frequencies gives an upward interval, which in most theories is used for further construction. So here: mFn -1 1 = F -1 -1 1 = $5/6 < 1$, and uFn 1 -1 = mFn 1 -1 = $6/5 = 1.2$ = a *minor third* upward.

The seventh partial tone corresponds to the factor 7 = F 0 0 0 1. It is the basis for “septimal” harmonic theories and music practice, as propagated by Martin Vogel and modelled by *Vogel's Tonnetz*. [14] The ratios to its non-octave predecessors in the harmonic series are the factors uFn -1 0 1 = $7/6 = 1.1\overline{6}$, called “septimal minor third”, and uFn 0 -1 1 = Fn 0 -1 1 = $7/5 = 1.4$, sometimes called the “natural tritone”, which is a very unsystematic name, because Fn 0 -1/3 1/3 is not known as a tone.

The ninth partial tone is not a prime, but the product of two stacked fifths. But it is interesting in relation to the root frequency and the just third: uFn 2 is the *greater tone* and uFn -2 1 is the *lesser tone* of just intonation, see below.

3.4 Compare the powers of the higher components of the harmonic series

The nearby meeting points of powers of fifth and octave motivate to look for similar effects involving the higher intervals. This means to search for some mF $\ldots a \ldots b \approx 1$, with $a, b \neq 0$. Indeed we find mFn 4 -1 = uFn 4 -1 = F -4 4 -1 = $81/80 = 1.0125$ = the *syntonic comma*: one just third is slightly smaller than four just fifths.

Furthermore, mFn 0 3 = F -7 0 3 = $125/128 = 0.9765625$ = three just major thirds are slightly smaller than an octave, the difference is called *lesser diesis*.

Then there is uFn -2 0 -1 = F -5 -2 0 -1 = $64/63 = 1.\overline{015873}$ = the *septimal comma*: One single just seventh downward (which can be considered one septimal whole tone upward) is slightly larger than two just fifths upward.

And finally uFn 1 0 0 1 = fF -5 1 0 0 1 = $33/32 = 1.03125 = 53.2728\ldots$ cent = the *undecimal comma*: The 11th partial tone is approximately a quarter tone higher than one just fourth up, which is one pure fifth down.

More complicated are the meetings of powers of more than one basis. The most prominent and relevant is that of the (just) minor third Fn 1 -1 and the octave. It holds that four minor thirds are slightly larger than an octave by mFn 4 -4 = F 3 4 -4 = $648/635 = 1.0368$ = the so-called *greater diesis*.

3.5 Common Western Tunings

A major scale relative to any root pitch and its frequency, for example

$$\langle \mathrm{C4}, \mathrm{D4}, \mathrm{E4}, \mathrm{F4}, \mathrm{G4}, \mathrm{A4}, \mathrm{B4}(,\mathrm{C5})\rangle$$

is constructed in the 12 EDO tuning,when setting $\mathrm{C4} \equiv 1$, by the sequence of intervals

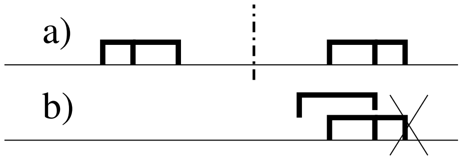

Figure 3. Two different procedures to invert a proportion.

$$\langle \mathsf{F}\,0, \mathsf{F}\,2/12, \mathsf{F}\,4/12, \mathsf{F}\,5/12, \mathsf{F}\,7/12, \mathsf{F}\,9/12, \mathsf{F}\,11/12\rangle$$

The Pythagorean construction for the same scale is

$$\langle \mathsf{uF}\,0, \mathsf{uFn}\,2, \mathsf{uFn}\,4, \mathsf{uFn}\,\text{-}1, \mathsf{uFn}\,1, \mathsf{uFn}\,3, \mathsf{uFn}\,5\rangle$$

The *just intonation* sees this scale as a compression of three stacked and overlapping major triads:

$$\mathrm{F} - \mathrm{A} - \mathrm{C} - \mathrm{E} - \mathrm{G} - \mathrm{B} - \mathrm{D}$$

It can be derived from the Pythagorean by replacing the thirds of these triads, the pitch classes A, E, and B, by the third as contained in the harmonic series. For these three chords we get

$$\langle \mathsf{uFn}\,\text{-}1, \mathsf{uFn}\,\text{-}1\;1, \mathsf{uFn}\,0, \mathsf{uFn}\,0\;1, \mathsf{uFn}\,1, \mathsf{uFn}\,1\;1, \mathsf{uFn}\,2\rangle$$

Arranged like the scale at the beginning of this section, we get

$$\langle \mathsf{uF}\,0, \mathsf{uFn}\,2, \mathsf{uFn}\,1\;1, \mathsf{uFn}\,\text{-}1, \mathsf{uFn}\,1, \mathsf{uFn}\,\text{-}1\;1, \mathsf{uFn}\,1\;1\rangle$$

Written as numbers this is

$$\langle 1, 9/8, 5/4, 4/3, 3/2, 5/3, 15/16\rangle$$

which is the same as

$$\langle 1, 1.125, 1.25, 1.\overline{3}, 1.5, 1.\overline{6}, 1.9375\rangle$$

The *derivation of the minor scale* can be done in different ways. When three points on some axis scale are given with two distinct inner distances, the inversion of the proportion can be done (a) by mirroring the whole structure by an arbitrary point of that axis, or (b) by adding the outer distance to the center point of the structure in one direction and discarding the outer point which lies in the other direction. Both procedures switch the order of the inner distances, see Figure 3.

The method (a) is used by the "dualistic" theories of harmonics (early Riemann, Oettingen, Karg-Ehlert), which use an "undertone spectrum" to derive the minor chord. The method (b) is used by the "monistic" theories of harmonics, where the major chords and modes are given, and from these the minors are derived, by re-arranging the patterns on the keyboard.

A monistic way to construct three minor chords and thus a minor scale, is to prepend to the triads above one factor which is a fifth $= \mathsf{uFn}\ 1$ lower than the third of the lowest triad, resulting in a new pitch D$*$. So we get the pitch classes of an a-minor scale as

$$\mathrm{D}^* - \mathrm{F} - \mathrm{A} - \mathrm{C} - \mathrm{E} - \mathrm{G} - \mathrm{B}$$

which is

$$\langle \mathsf{Fn}\ \text{-}2\;1, \mathsf{Fn}\ \text{-}1, \mathsf{Fn}\ \text{-}1\;1, \mathsf{F}\,0, \mathsf{Fn}\ 0\;1, \mathsf{Fn}\ 1, \mathsf{Fn}\ 1\;1\rangle$$

This shows clearly that the common textbook sentence "a-minor and C-major use the same set of pitch classes" indeed depends on the applied tuning system: Here the two pitches called D and D* differ by the syntonic comma.

$$\widehat{\mathsf{c}}\_ : \mathbb{N} \to (\mathbb{R} \leftrightarrow \mathbb{R})$$
$$\widehat{\mathsf{c}}_p = \{x, y : \mathbb{R} \mid \exists z : \mathbb{Z} \bullet x \cdot p^z = y\}$$

$$\widehat{\mathsf{u}}\_, \widehat{\mathsf{m}}\_ : \mathbb{N} \to (\mathbb{R} \leftrightarrow \mathbb{R})$$
$$\widehat{\mathsf{u}}_p = \lambda x : \mathbb{R} \mid 1 \le x < p \bullet x$$
$$\widehat{\mathsf{m}}_p = \lambda x : \mathbb{R} \mid 1/\sqrt{p} < x \le \sqrt{p} \bullet x$$

$$\mathsf{c}_p = \mathsf{f}^{\sim} \circ \widehat{\mathsf{c}}_p \circ \mathsf{f}$$
$$\mathsf{u}_p = \mathsf{f}^{\sim} \circ \widehat{\mathsf{u}}_p \circ \mathsf{f}$$
$$\mathsf{m}_p = \mathsf{f}^{\sim} \circ \widehat{\mathsf{m}}_p \circ \mathsf{f}$$

Table 7. A variant of F-operator for systems without octave identity.

The *quarter-comma meantone* tuning can most naturally be derived as follows: When tuning for instance a harpsichord by stacking pure fifths from C2, one reaches with E4 nearly the fifth partial tone of C2. So if we want this partial tone to sound on this key, we restart the whole process: Instead of pure fifths we use use the fourth part of its frequency factor $= \mathsf{F}\ 0\ 0\ 1/4 = \mathsf{uF}\ 0\ 0\ 1/4$. sometimes called "meantone fifth".

The intonation process follows the spiral of fifths as usual. As with $\mathsf{uF}\,0\,1$, it never comes back to a frequency class it has passed. The quarter-comma meantone tuning thus defines a scale of infinite many frequencies—the historical keyboard tunings realize a segment of twelve of these. The most frequently found keyboard design goes from E♭, B♭, F, ... up to C♯, G♯. With C$\equiv 1$, these frequency classes are

$$\langle \mathsf{uFn}\,0\ \text{-}3/4, \mathsf{uFn}\,0\ \text{-}1/2, \mathsf{uFn}\,0\ -1/4, \mathsf{cF}\,0,$$
$$\mathsf{uFn}\,0\;1/4, \mathsf{cF}\,0\;1/2, \mathsf{uFn}\,0\;3/4, \mathsf{uFn}\,0\;1,$$
$$\mathsf{uFn}\,0\;5/4, \mathsf{uFn}\,0\;3/2, \mathsf{uFn}\,0\;7/4, \mathsf{uFn}\,0\;2\rangle$$

The *wolf fifth* goes from the last of these to the first: $\mathsf{u}(\mathsf{F}\ 0\,0\ -3/4\ /\ \mathsf{F}\ 0\,0\,2) = \mathsf{uF}\ 0\,0\ \text{-}11/4 = 1.1993... \approx 738\,\mathsf{cent}$. It is obviously different from both $\mathsf{uFn}\ 1 = 1.5 \approx 702\,\mathsf{cent}$ and $\mathsf{uFn}\ 0\,1/4 = 1.4953... \approx 697\,\mathsf{cent}$.

The construction of the *third-comma meantone* tuning is one step more indirect: When stacking three pure fifths we come very near to the frequency $\mathsf{F}\ 1\ \text{-}1\ 1$, which is a pure fifth lower than the pure third and its class can be called a *minor third* below C (or a major sixth above). So we get $\mathsf{F}\ 1/3\ \text{-}1/3\ 1/3$ as the *third-comma meantone fifth*.

Werckmeister III forces the spiral of fifths into a circle by subtracting a quarter of the Pythagorean comma $= \mathsf{F}\ 19/4\ 3$ from the fifths over C, G, D, and E. The result (with C $\equiv 1$):

$$\langle \mathsf{cF}\,0, \mathsf{cF}\ \text{-}19/4\ \text{-}2, \mathsf{cF}\ \text{-}19/2\ \text{-}4, \mathsf{cF}\ \text{-}57/4\ \text{-}6,$$
$$\mathsf{cF}\ \text{-}57/4\ \text{-}5, \mathsf{cF}\ \text{-}19\ \text{-}7 = \mathsf{cFn}\ \text{-}7, \mathsf{cFn}\ \text{-}6, \mathsf{cFn}\ \text{-}5,$$
$$\mathsf{cFn}\ \text{-}4, \mathsf{cFn}\ \text{-}3, \mathsf{cFn}\ \text{-}2, \mathsf{cFn}\ \text{-}1\rangle$$

3.6 Rare Tunings

Renold et.al. [3, p. 95] define a tuning from an anthroposophic background. They consider the equal division of the octave as a consonance. Their tuning is Pythagorean (with pure fifths) for the black and white piano keys separately, both groups connected by $\sqrt{2}$ between C and F♯. Intervals between black and white keys are called "geformt" ("formed"). The chromatic scale on C thus reads

$$\langle \mathsf{uF}\,0, \mathsf{uF}\,1/2\;1, \mathsf{uFn}\ 2, \mathsf{uF}\,1/2\;3, \mathsf{uFn}\ 4, \mathsf{uFn}\ \text{-}1,$$
$$\mathsf{uF}\,1/2, \mathsf{uFn}\ 1, \mathsf{uF}\,1/2\;2, \mathsf{uFn}\ 3, \mathsf{uF}\,1/2\;4, \mathsf{uFn}\ 5\rangle$$

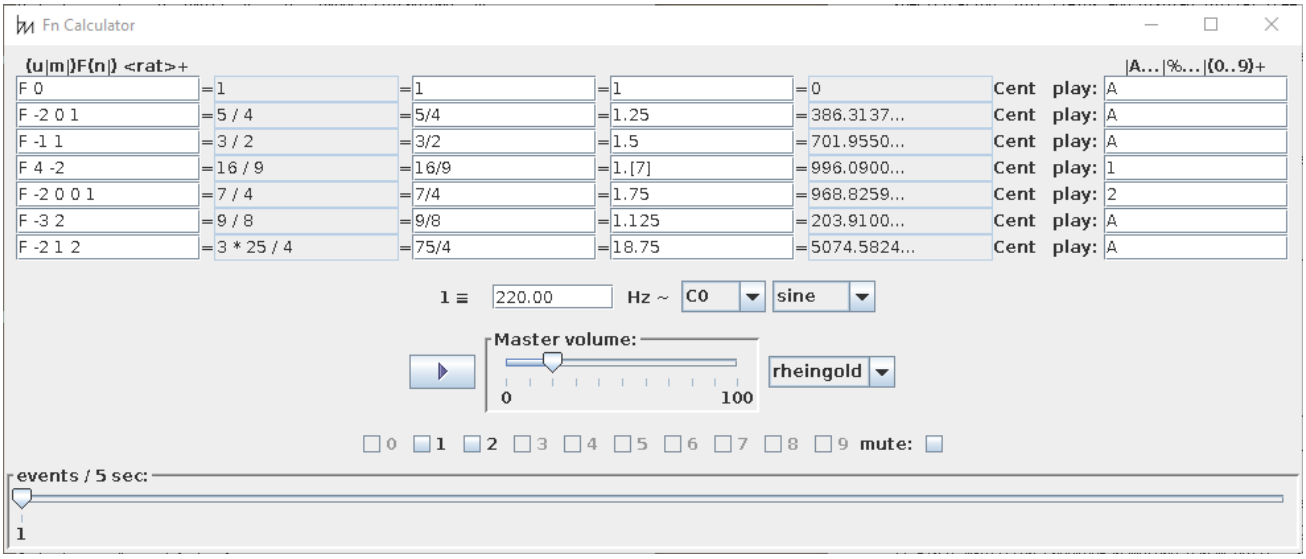

Figure 4. Screenshot of the application to explore F-operator expressions.

3.7 Symbolic calculi not completely compatible with F-operator

The family of *Bohlen–Pierce scales*, and any other system not employing the identity modulo octave, is not fully compatible with all components of F-operator.

A *just* Bohlen–Pierce scale can be described by

$$\langle \mathsf{F}\,0, \mathsf{Fn}\ -1\,2\,-1, \mathsf{Fn}\ 2\,0\,-1, \mathsf{Fn}\ 0\,-1\,1, \mathsf{Fn}\ -1\,1, \mathsf{Fn}\ 2\,-1, \mathsf{Fn}\ 1\,1\,-1, \mathsf{Fn}\ -1\,0\,1, \mathsf{Fn}\ -2\,2, \mathsf{Fn}\ 1\rangle$$

The intervals between the steps of this scale are easily calculated as the series of differences:

$$\langle \mathsf{Fn}\ -1\,2\,-1, \mathsf{Fn}\ 3\,-2, \mathsf{Fn}\ -2\,-1\,2, \mathsf{Fn}\ -1\,2\,-1, \mathsf{Fn}\ 3\,-2, \mathsf{Fn}\ -1\,2\,-1, \mathsf{Fn}\ -2\,-1\,2, \mathsf{Fn}\ -1\,2\,-1, \mathsf{Fn}\ 3\,-2\rangle$$

How conveniently the F-operator notation can be used to *construct* these scales is still to be explored.

The operators cF, uF, and mF cannot be applied, because they work modulo octave. Table 7 shows more general definitions of the fundamental operators—further definitions can be adopted accordingly.

Any *tempered* Bohlen–Pierce scale can be described like

$$\langle \mathsf{F}\,0, \mathsf{Fn}\ 2/13, \mathsf{Fn}\ 3/13, \mathsf{Fn}\ 4/13, \mathsf{Fn}\ 6/13, \mathsf{Fn}\ 7/13, \mathsf{Fn}\ 9/13, \mathsf{Fn}\ 10/13, \mathsf{Fn}\ 12/13\rangle$$

4. IMPLEMENTATION

An irrational number which is an instance of $\widehat{\mathbf{F}}$ can thus be precisely represented in a computer. This is employed by the demo software, see figure 4. The binary is found at `http://bandm.eu/downloads/FnCalc.jar`; the sources shall be published in near future.

Each line stands for one particular factor: The first column allows the input of all F-operator expressions types. These will be normalized to the explicit form "F...". The second column shows a "fraction of products of powers", while the third column shows the fractional value, if it is rational. This column can also be used for input. The next column shows the decimal fraction representation, either exact for rational values or only a prefix, as usual for irrational values. Rational values can be entered. Then follows the output of the cent value. Whenever a value is entered, all other columns are updated accordingly, or the entry field is colored red to indicate a syntax error.

The right half of the GUI makes the factors audible, relative to a user-defined reference frequency: They can be switched on individually and can be arranged to chords, which can be played by a primitive sequencer.

One selection menu is for the kind of sound used for all chords, another initializes one of several demo set-ups.

5. RELATED WORK

Independent and earlier are the proposals by Joe Monzo [9, 10]. Main deficiencies: they have never been scientifically published ("my book" mentioned in [10] is unavailable), the necessary mathematics are not fully developed, and the functions of F-operator beyond the basic "F" are missing. Nevertheless the fundamental idea is very similar, and we gratefully borrow the trick to use simple whitespace as separators for the F-operator abbreviated format.

6. REPHRASING RELEVANT WORKS FROM LITERATURE

6.1 Examples from Wikipedia

All entries in the German Wikipedia related to intervals, tuning, harmonic theory, etc., suffer from the fact that many contributors want to present their personal opinion as mandatory. But indeed in all these fields there are multiple methods, more or less oriented towards practice or towards theory, and a lexicon is hardly suited to present them all, with equal rights and cleanly separated.

As a consequence, we find an overwhelming amount of redundancy, in three concerns: (a) different users "occupy"

In music theory and tuning, the kleisma (Greek: $\kappa\lambda\epsilon\iota\sigma\mu\alpha$), or semicomma majeur, is a minute and barely perceptible comma type interval important to musical temperaments. It is the difference between six justly tuned minor thirds F **1 1 -1** (each with a frequency ratio of 6/5) and one justly tuned tritave or perfect twelfth F **0 1** (with a frequency ratio of 3/1, formed by a 2/1 octave plus a 3/2 perfect fifth). It is equal to a frequency ratio of F **-6 -5 6** $= 15625/15552 = 2^{-6} * 3^{-5} * 5^{6}$, or approximately 8.1 cents. It can be also defined as the difference between five justly tuned minor thirds $(\textbf{F 5 5 -5})^{-1}$ and one justly tuned major tenth ***** F**-1 0 1** (of size 5/2, formed by a 2/1 octave plus a 5/4 major third) or as the difference between a chromatic semitone **=** F **-3 -1 2** $(= 25/24)$ and a greater diesis **/** F**3 4 -4** $(= 648/625)$.

Table 8. Wikipedia EN, lemma "Kleisma". `https://en.wikipedia.org/wiki/Kleisma`[20251225]

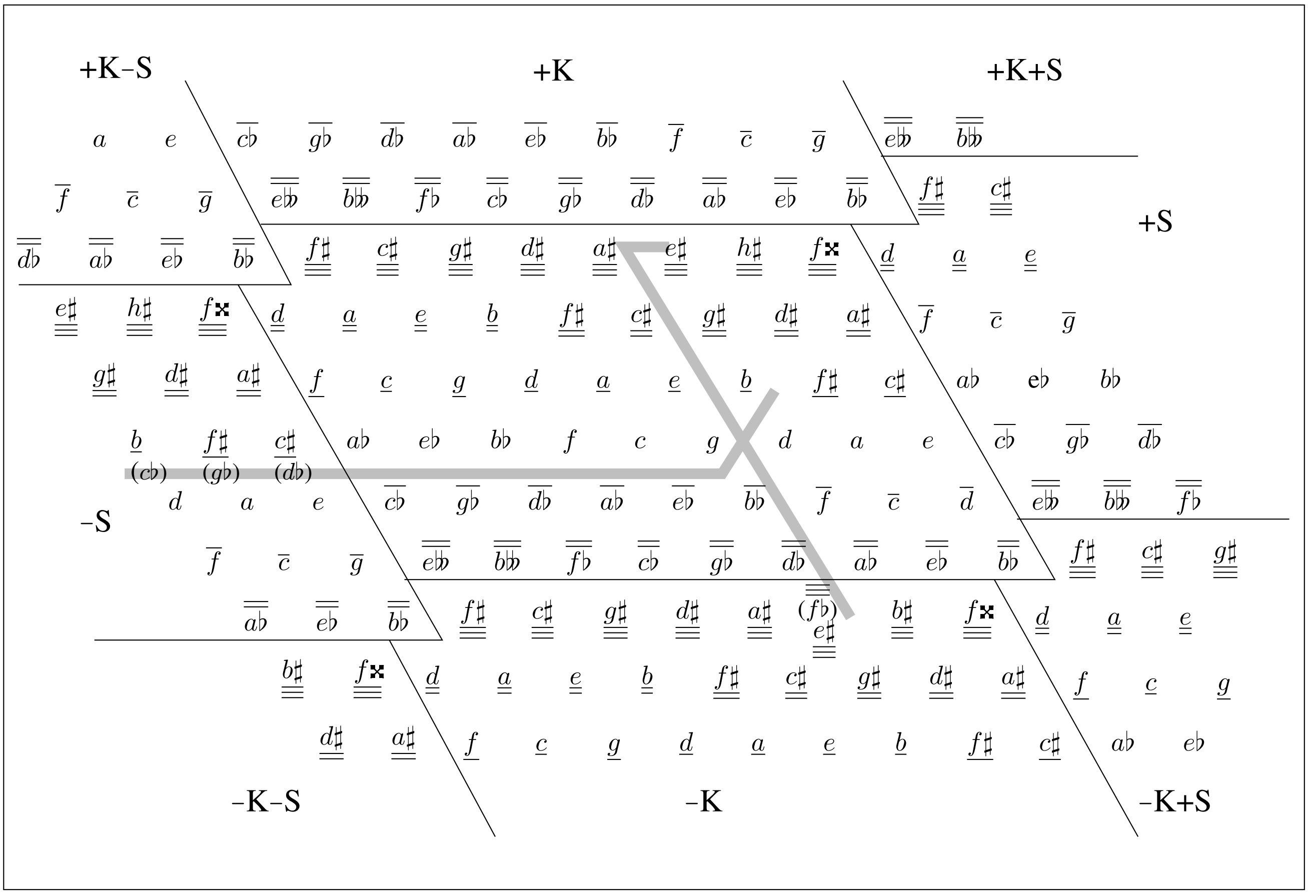

Figure 5. The original figure from [15], translated to English pitch class names. The graphic representation of the first two calculations from Table 9, in gray and in parentheses, added by us. $\underline{a}$ means one syntonic commata lower than a, etc.

different lemmata to apply different theories, (b) different names for the same arithmetic intervals are used, and (c) the same arithmetic expressions appear again and again, in many different renderings.

Table 8 shows the begin of the text for the lemma "kleisma" in the English wikipedia. Inserted in bold print are the F-operator notations, and outgrayed everything redundant. This includes the complete last sentence, which becomes well-neigh absurd: The F-operator notation makes obvious that infinite many other constructions of this particular interval are possible. (What presumably is meant here are different concrete *construction processes* which all lead to an interval of this size. But these should perhaps be better described elsewhere.) The wording "difference between a and b" is often unclear whether to divide a/b or vice versa, see the operators required in the last sentence in Table 8.

6.2 The 53 EDO tuning according to Tanaka

An early derivation of 53 EDO was by Nicolaus Mercator, according to [2, p. 531], and even earlier by Jing Fang 40 BCE., according to [13, p. 208]. Both were led by a Pythagorean approach, namely the search for near meeting points of the serieses of fifths and octaves.

Contrarily, the derivation by Tanaka [15] starts with *just* intonation, i.e. it searches for approximations of both uFn 1 and uFn 0 1. Our text in Table 9 rephrases that section of the famous article. The reader will need not much German to judge the gain in compactness.

6.3 "Musical Scales of the World" by Hewitt

Hewitt's "Musical Scales of the World" [12] addresses pop musicians (in the widest sense) and music enthusiasts. This

[page 8: “Zunächst berechnen wir das Intervall . . .]
First we calculate the interval between $\underline{b}$ and $c\flat$: $\underline{b}$ is the pure major third of g, so $\underline{b}/g = \mathsf{uFn}\ 01$.
You get $c\flat$ by eight steps of a fourth:

$$g - c - f - b\flat - e\flat - a\flat - d\flat - g\flat - c\flat$$

so $c\flat/g = \mathsf{uFn}\ -8$ and $\underline{b}/c\flat = \mathsf{uFn}\ 8\ 1 = \mathsf{F}\ \text{-}15\ 8\ 1 = 32805/32768 = S \approx 887/856$.
This interval, called “schisma”, is according to W. Freyer already beyond perceptive distinction. [. . .] So we get the following equivalences (with uFn as distances from g):

$\underline{b} = \mathsf{uFn}\, 0\ 1 \equiv c\flat = \mathsf{uFn}\, -8$ and $\underline{\underline{a\sharp}} = \mathsf{uFn}\, 1\ 2 \equiv \underline{b\flat} = \mathsf{uFn}\, -7\ 1$ and $f\sharp = \mathsf{uFn}\, 5 \equiv \overline{g\flat}$ [sic!] $= \mathsf{uFn}\, -3 - 1$ *(etc. . .)*

[page 9: “Ferner wollen wir das Intervall. . . ”]
Furthermore, we want to compare a tone from the diagonal row of steps of thirds with a nearest tone from the neighboring diagonal. Let us compare $\overline{\overline{\overline{f\flat}}}$ and $\underline{\underline{\underline{e\sharp}}}$.
From $\overline{\overline{\overline{f\flat}}}$ to $\underline{\underline{\underline{a\sharp}}}$ downwards by minor thirds is

$$\overline{\overline{\overline{f\flat}}} - \overline{\overline{d\flat}} - \overline{b\flat} - g - \underline{e} - \underline{\underline{c\sharp}} - \underline{\underline{\underline{a\sharp}}}$$

We get the factor $\overline{\overline{\overline{a\sharp}}}/\underline{\underline{\underline{f\flat}}} = \mathsf{uFn}\ \text{-}6\ 6$. But $a\sharp/e\sharp = \mathsf{uFn}\ \text{-}1 = 3/4$.
Thus $\underline{\underline{\underline{e\sharp}}}/\overline{\overline{\overline{f\flat}}} = \mathsf{uFn}\ \text{-}6\ 6 * \mathsf{uFn}\ \text{-}1 = \mathsf{uFn}\ \text{-}5\ 6 = \mathsf{F}\ \text{-}6\ \text{-}5\ 6 = 15625/15552 = 1.004694.. = K \approx 214/213$.
This interval I want to call “kleisma”, which in higher ranges can only be heard with dedicated concentration. One can therefore set without losing the tuning (uFn again measured from g):

$$\overline{\overline{\overline{g\flat}}} = \mathsf{uFn}\ 5\ \text{-}3 \equiv \underline{\underline{\underline{f\text{𝄪}}}} = \mathsf{uFn}\ 3 \ \text{ and } \ \overline{\overline{b\flat}} = \mathsf{uFn}\ 5\ \text{-}2 \equiv \underline{\underline{\underline{a\text{𝄪}}}} = \mathsf{uFn}\ 0\ 4 \ \text{ etc.}$$

[page 10: “Durch Anwendung der schismatischen und kleismatischen Verwechselung . . . ”] By applying the schismatic and kleismatic changes, we can reduce the number of distinct tones to 53, see Figure 5.

[page 15: “Von einem Tone ausgehend . . . ”]
Starting with a particular tone, go three fifths to the right and a minor third down, i.e. add $\mathfrak{K} = \mathsf{uFn}\, 4\ \text{-}1$. You reach a tone which is a syntonic comma sharper than the start point. Repeat this until a cycle is closed. Whenever you cross the border into the neighboring parallelogram which is labeled $+K$, this indicates that all pitches here are one K sharper than in the main parallelogram. Into this we jump back, so we execute a smaller step than a normal $\mathfrak{K}$, so we have to subtract K from the reached frequency. Correspondingly for $-S$, etc.
For instance, start from $c = \mathsf{uFn}\, 0$, follow three fifths to the right to $a = \mathsf{uFn}\, 3$ and then a minor third up to $\overline{c} = \mathsf{uFn}\, 4\ \text{-}1$. The next such step goes in fifths over the border to $\overline{\overline{\overline{b\flat\flat}}}$ and over the second border to $\underline{\underline{\underline{c\sharp}}}$. The tone $\underline{\underline{\underline{c\sharp}}}$ is found again in the diagram, in its top left corner, and we jump there. This calculates to $\mathsf{u}(\mathsf{uFn}\, 4\ \text{-}1 * \mathfrak{K} * K/S) = \mathsf{u}(\mathsf{uFn}\, 4\ \text{-}1 * \mathsf{uFn}\, 4\ \text{-}1 * \mathsf{uFn}\ \text{-}5\ 6/\, \mathsf{uFn}\, 8\ 1) = \mathsf{uFn}\ \text{-}5\ 3$
From there we can continue to $\underline{\underline{c\sharp}} = \mathsf{uFn}\ \text{-}1\ 2$ without further border crossings.
Every single step in a complete cycle spans either an altered or a normal syntonic comma, whether or not a border is crossed:

- twelve steps cross only the right border and thus include one schismatic change. Its size can be labelled $\mathfrak{K}/S = \mathsf{uFn}\ 4\ \text{-}1/\, \mathsf{uFn}\ 8\ 1 = \mathsf{uFn}\ \text{-}4\ \text{-}2$.
- five steps cross only the lower border and thus contain a kleismatic change. Their size is $\mathfrak{K} * K = \mathsf{uFn}\ 4\ \text{-}1 * \mathsf{uFn}\ \text{-}5\ 6 = \mathsf{uFn}\ \text{-}1\ 5$.
- seven steps contain both changes, of size $\mathfrak{K} * K/S = \mathsf{uFn}\, 4\ \text{-}1 * \mathsf{uFn}\ \text{-}5\ 6/\, \mathsf{uFn}\, 8\ 1 = \mathsf{uFn}\ 7\ 4$.
- 29 steps of unaltered syntonic commata of size $\mathfrak{K}$.

By executing this whole cycle of 53 steps of (normal and altered) syntonic commata, we finally reach $\mathfrak{K}^{29} * (\mathfrak{K}^{12}/S^{12}) * (\mathfrak{K}^5 * K^5) * (\mathfrak{K}^7 * K^7/S^7) = \mathfrak{K}^{53} * K^{12}/S^{19} = \mathsf{F}\ \text{-}212\ 212\ \text{-}53 * \mathsf{F}\ \text{-}72\ \text{-}60\ 72/\, \mathsf{F}\ \text{-}285\ 152\ 19 = \mathsf{F}\ 1 = 2$.
It follows that $k = 2/\mathfrak{K}^{53} = K^{12}/S^{19} = \mathsf{F}\ \text{-}72\ \text{-}60\ 72\ /\ \mathsf{F}\ \text{-}285\ 152\ 19 = \mathsf{F}\ 213\ \text{-}212\ 53 = 1.03536.. \approx 29/28$ is the factor by which a sequence of 53 unaltered syntonic commata is too small to close the octave.
So we can close the cycle by a modified step width of $(\mathsf{uFn}\, 4\ \text{-}1) * \sqrt[53]{k}$.
This division can much more directly be realized by F 1/53, as practiced since long (Didymus(?), Kircher, N. Merkator).
But with our derivation, the approximation of the just third $\mathsf{F}(17/53) \approx \mathsf{uFn}\ 0\ 1$ is not merely accidental but intended.

Table 9. Rephrasing of the derivation of 53 EDO starting from just intonation according to Tanaka [15]. See Figure 5.

is supported by guitar fret diagrams and keyboard diagrams, given for every scale. Its intentions are doubtlessly laudable. But it is questionable to quote scale constructions from overall of the world and transfer them with only little comment to the 12 EDO keyboard.

Late in the text, Hewitt confesses "[In case of] Indonesian Gamelan, [...] Thai, [...] and African [...] music, tuning is more or less an indispensable feature of the sound. Consequently to try and bring these scales over and adapt them to the Western scale is often to lose a great part of their original charm." [section 9, p. 140]

But this applies also to all other scales, for which the underlying tuning is merely given by a label, like "Tuning: Twelve-tone equal temperament" or "Tuning: Just". which may be ambiguous with particular scales, or even less unspecific: "Tuning: By ear" or "Tuning: Varies."

Already for the first third of the book, the more or less conventional Western scales, the tuning indication is often well-neigh false: It must be doubted that a solo violinist playing in "Hungarian minor I" uses 12 EDO intonation—the same with Gregorian singers using Dorian mode.

Only sections 9 to 11 additionally contain cent values for the reader to tune their synthesizer. For gamelan music, he quotes *measurements* carried out by Ellis, McPhee, Kunst, and others, see also p. 159. But in the context of African "Silaba" and "Tomora Masengo" [p. 165*f.*] just intervals (empirically measured) are in the same scale with cent values 500 and 700 (obviously from 12 EDO).

We suppose that this violates the fundamental difference between cent values coming from measured physical reality, and tuning systems, which are *concepts* and can be found by interviewing musicians and instrument makers.

The merits of this book would be much larger when the scales had been given in their true, intended structure, throughout. To *approximate* this by a 12 EDO keyboard is a subsequent step. That the first step has been skipped is understandable w.r.t. the addressed audience, but nevertheless a regrettable methodological flaw.

For all intended structures, the F-operator system would have been a better choice for presentation, as shown in table 10 for some examples. They start with the acoustic scale, because there the criticized deficiency is already indicated by its name. The laconic label "tuning: just" is for many scale degrees ambiguous (see "Bilawal Thãt", p. 161, and "Mela Naganandini", p. 209) and also can be clarified by simple F-operator expressions.

6.4 Free combination of tuning systems in *bach* for Max

The *bach* system is a very elaborate library for compositional operations (in a traditional sense of the word) in the context of the Max framework for processing. Agostini and Ghisi [16] present an extension of the pitch notation in this library, suited to support *just tuning*.

Before this extension, the notation was already overloaded: each English term for a pitch (i.e. a pitch class name followed by the number of an octave register) stands either for the frequency of this pitch, or for the factor of this frequency relative to that of the pitch "`C0`". These frequencies and factors are calculated by applying the 12 EDO tuning to the pitch names.

A new additional format for pitches is now written with an inserted pair of braces, like "`C{}4`". These expressions stand for the frequencies and factors resulting from *Pythagorean* tuning, following the spiral of fifths.

To describe scales in *pure tuning in all keys*, the count of the *commata* to apply is inserted into the braces. They are identified by their position, separated by semicola, and trailing zeroes are of course implicit. Using explicit commata is a very established method, see for instance Helmholtz [2, p. 453] for the syntonic comma. It has the disadvantage that their direction is not evident without re-calculating their construction—does the "11-comma" go up or down in the frequency world? Indeed, using commata to define pitches is a *detour*: First we must notate what we do *not* mean, and then apply the necessary correction.

The big advantage realized by this extension to *bach* is its *compositionality*: Both kinds of frequencies and factors can freely be mixed in all arithmetic operations. Agostini and Ghisi point out that a 12-tone-tempered frequency can be added to and subtracted from any just interval, etc.

Indeed this can have practical relevance: Hitting an `E2` fortissimo on a grand piano, and then filtering the resulting sound for its fifth partial tone needs a filter frequency which is such a mixture. In *bach* that task can now be written as `E2+E{-1}2`: Start with the 12 EDO tone `E2`, add four just fifths (which happens to be a Pythagorean major third plus two octaves, thus written `E{}2`), and then go down one syntonic comma again, because you don't want the Pythagorean but the just third. Together with the two octaves, this third reaches the fifth partial tone. In F-operator we can rephrase as $\mathsf{F}(2 + 4/12) * \mathsf{F}\ \ 0\ \ 0\ \ 1 = \mathsf{F}\ \ 7/3\ 0\ 1$.

The presentation of paper [16] on the Tenor 2025 conference has motivated the development of F-operator.

7. ACKNOWLEDGEMENTS

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Section 3: Synthetic Modes and Scales							
acoustic	uF 0	uFn 2	uFn0 1	uFn 11 ↑ 1	uFn 1	uFn 13 ↑ 1	uFn 0 0 1
Hungarian minor I	uF 0	uFn 2	uFn1 −1	uF2 1	uFn 1	uFn 0 −1	uFn 2 −1
						(or	uFn 0 0 1 ?)
Hungarian minor II	uF 0	uFn 2	uFn1 −1	uF2 1	uFn 1	uFn 0 −1	uFn 1 1
Double harmonic	uF 0	uFn −1 −1	uFn0 1	uF−1	uFn 1	uFn 0 −1	uFn 1 1
Section 7: Pentatonic Scales of Japan							
Ryo	uF 0	uFn 2	uFn 4	uF 1	uFn 3		
Ritsu	uF 0	uFn 2	uFn −1	uF 1	uFn 3		
Minyõ	uF 0	uFn −3	uFn −1	uF 1	uFn −2		
Hirajoshi	uF 0	uFn 2	uFn −3	uF 1	uFn −4		
Iwato	uF 0	uFn −5	uFn −1	uF −6	uFn −2		
Kokin Joshi	uF 0	uFn 2	uFn −1	uF 1	uFn −4		
Akebono	uF 0	uFn 2	uFn −3	uF 1	uFn 3		
Yosenpo	uF 0	uFn 2	uFn −1	uF 1	↗uFn 4	↘uFn −2	
Insenpo	uF 0	uFn −5	uFn −1	uFn 1	↗uFn −2	↘uFn −4	
Section 9: Indonesian Scales of the Gamelan							
Slendro	uF 0	uF 1/5	uF 2/5	uF 3/5	uF4/5		
Section 10: Equatonal Scales of Thailand							
Thai seven-toned	uF 0	uF 1/7	uF 2/7	uF 3/7	uF 4/7	uF 5/7	uF 6/7
Thai mode 1	uF 0	uF 1/7	uF 2/7		uF 4/7	uF 5/7	
Thai mode 3	uF 0	uF 1/7		uF 3/7	uF 4/7		uF 6/7
Section 12: Scales of Northern India							
Khamaj Thãt	uFn 0	uFn 2	uFn 0 1	uFn −1	uFn 1	uFn 3	uFn−2
	(or	uFn −2 1				uFn −1 2	uFn 2 −1 ?)
Section 13: Scales of Carnatic Music							
Mela Naganandini	uFn 0	uFn 2	uFn 0 1	uFn −1	uFn1	uFn 2 2	uFn 1 1
					(or	uFn 6 1 ?)	

Table 10. Some proposals to enhance "Musical Scales of the World" [12] by substantial tuning information.

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A. MATHEMATICAL NOTATION

The employed mathematical notation is fairly standard, inspired by the Z notation [17]. The following table lists some details:

$\mathbb{N}$	All natural numbers, incl. Zero.
$\mathbb{Z}, \mathbb{Q}$	All integer and rational numbers, resp.
$A \times B$	The product type of two sets A and B.
$A \leftrightarrow B$	The type of the relations between A and B.
$A \rightarrow B, A \nrightarrow B$	The total and partial functions from A to B.
$r^{\sim}$	The inverse of a relation
$f \llparenthesis s \rrparenthesis$	The image of the set s under the relation f
$r \circ s$	The composition of two relations.
$\mathsf{dom}\ A, \mathsf{ran}\ A$	Domain and range of a relation.
$r \oplus s$	Overriding of function or relation r by s.
$R \mathbin{\rhd\!\!\!-} S$	$= R \setminus (\mathsf{dom}\ R \times S)$, i.e. negative range restriction of a relation.
$\mathsf{SEQ}\ A$	Lists as maps $\{0..n\} \rightarrow A$.
$\langle \ldots \rangle$	Instance of a list type.

Functions are considered as special relations; relations as sets of pairs. So with functions, expressions like "$f \cup g$" are defined.